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Prices of Production are Proportional to Real Costs

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Prices of production are proportional to real costs[1]

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Abstract

I take a 'circular flow', monetary-production approach to simple reproduction in a state of self-replacing equilibrium. The main result is that profit-equalising prices of production are proportional to real costs. The principle of real cost is hidden in Sraffa's 'surplus' approach, which instead directs attention to the role of prices as indices of distribution. Yet the two approaches are formally equivalent. They are merely different representations of the same theory of prices of production.

Hence, contrary to classical and modern authors, equilibrium prices of production do not diverge from labour values due to profit-rate equalisation. Marx's theoretical insight that labour-cost is conserved in price is correct. In Sraffa's system, the existence of a constant of proportionality, the monetary expression of labour-time, such that total value equals total price and total surplus-value equals total profit, is a theorem. The mismatch between the dual systems of value and price, traditionally known as the transformation problem, is in fact a real-cost accounting error due to the omission of the labour-cost of money-capital.

The analysis is restricted to single-product linear production systems in a state of self-replacing equilibrium.

Contents

1. Introduction

The publication of Sraffa's *Production of Commodities by Means of Commodities* (PCMC) in 1960 initiated a renaissance in the classical approach to value and income distribution [37]. Sraffa's theoretical insights were subsequently employed to critique the neoclassical or marginal theory of value [10] and the Marxist labour theory of value [77, 42]. Theories of economic value try to identify price-independent parameters that are the ultimate source or cause of price [9]. But according to the Sraffian or 'neo-Ricardian' critiques of value theories there is no determinate relationship between either marginal productivities or labour-time and prices. Some interpreters [53, 73] of Sraffa argue that his work demonstrates that the search for an essence of value, some single substance or principle that explains the phenomena of prices, is redundant, once it is realised that 'prices are simply implicated in a given structure of production and distribution and *nothing is hidden behind them*' [73]. Often deep scientific progress consists in conceptual revolution. Perhaps the search for a theory of value is founded on a conceptual error, of similar status, say, to the defunct 17th century concept of phlogiston, a substance thought to be liberated upon combustion.

The argument of this paper is that nihilistic conclusions regarding the theory of economic value are misguided because they are ultimately based on Sraffa's failure to consider that money-capital is a commodity with an associated cost of production. A purely nominal treatment of income distribution obscures an important value-theoretic principle. Sraffian prices, in the case of single production, have a hitherto unnoticed proportional relationship to particular kinds of scalar quantities, real costs, which can be measured according to any chosen units of quantities, be it labour-time, bushels of corn, or tonnes of iron. The important yet very simple *principle of real-cost* is 'hidden behind' Sraffa's prices of production. A corollary of this result is a solution to the transformation problem that reinstates the quantitative connections between labour-time and price, a relationship Sraffa thought obtained only in the special case of zero profits. Hence, Sraffa's theoretical insights, contrary to the prevailing interpretation, provide the basis for the logical completion of the classical labour theory of value, rather than its critique.

This paper is split into two dependent parts. Part 1 derives the value-theoretic principle of real-cost from a critique of Sraffa's 'surplus' representation of simple reproduction. Part 2 applies the principle of real-cost to the neo-Ricardian critique of Marx's labour theory of value.

Part 1. The principle of real-cost

2. Sraffa's system

Sraffa employs a system of simultaneous linear equations to model the input-output relations of an economy. His approach belongs to a family of linear production models that includes Leontief's closed and open models, and Von Neumann's model of a growing economy (see [56, 36] for a discussion). The following summary of Sraffa's system is more formal than the original and differs in some small details.

In Sraffa's single production model, each industry or sector produces a single commodity by means of others. There is no fixed capital. This is a static situation, so the scale of production does not change, and therefore Sraffa makes no assumptions regarding returns to scale. The technical methods of production are represented by a matrix of inter-activity coefficients and a vector of labour coefficients.

Definition 2.1. The *technique* of an economy is a non-negative $n \times n$ matrix of inter-activity coefficients, $\mathbf{A} = [a_{i,j}]$. Each $a_{i,j} \geq 0$ represents the physical quantity of commodity-type i directly required to output 1 unit of commodity-type j.

Definition 2.2. The *direct labour coefficients* are a $1 \times n$ row vector, $\mathbf{l} = [l_i]$. Each $l_i > 0$ represents the physical quantity of labour-power directly required to output 1 unit of commodity-type i.

The technique and labour coefficients define a production graph that describes the input-output relationships that obtain in the economy. Figure 1 depicts an example production graph.

Given an input-output economy we are normally interested in the quantities of commodities produced and their prices.

Definition 2.3. The *net product*, or surplus, is a $1 \times n$ row vector, $\mathbf{n} = [n_i]$. Each $n_i \geq 0$ represents the physical quantity of commodity-type i that is available for consumption after capital stocks have been replaced.

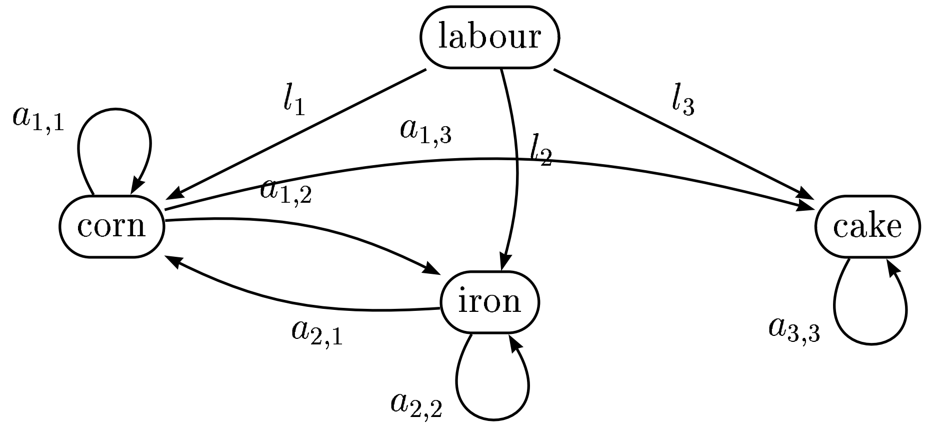
Definition 2.4. A *Sraffian quantity equation* is

$$\mathbf{q} = \mathbf{q}\mathbf{A}^{\mathrm{T}} + \mathbf{n}, \tag{1}$$

where $\mathbf{q} = [q_i]$ is a $1 \times n$ row vector of quantities. Each $q_i > 0$ is measured in units of commodity-type i.

Equation (1) defines equilibrium quantities. Consider a production period. At the end of the period there is a gross output of $\mathbf{q}$ commodities. A stock of input commodities, $\mathbf{q}\mathbf{A}^{\mathrm{T}}$, was used-up and replaced during the period. The economy produces more than it uses up leaving a net product, or net national income, or surplus, $\mathbf{n}$, which is available for consumption. Production transforms a set of commodity inputs and produces a larger output. A part of the output replaces the inputs; the remainder is consumed as surplus.

Sraffa restricts his analysis to viable economies able to produce a surplus. For the surplus to be positive the economy must be productive.



$$[\mathbf{A}, \mathbf{l}]^{\mathrm{T}} = \begin{array}{c} \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & 0 \\ 0 & 0 & a_{3,3} \end{bmatrix} \\ \begin{bmatrix} l_1 & l_2 & l_3 \end{bmatrix} \end{array}.$$

FIGURE 1. **Production graph for a 3-sector economy**. The graph visualises the direct input requirements for the production of 1 unit of output. For example, 1 unit of corn directly requires $a_{1,1}$ units of corn, $a_{2,1}$ units of iron, and l_1 units of labour for its production.

Definition 2.5. A technique is *productive* if there is a column vector $\mathbf{x} \in \mathbb{R}^n_+$ such that $\mathbf{x} > \mathbf{A}\mathbf{x}$.

This condition ensures that it is possible that the economy can produce a surplus in all commodities (c.f. Bidard's definition of strict viability [5]).

Sraffa assumes that labour is homogeneous and the wage rate and profit rate are uniform over the whole economy. Both assumptions can be viewed as non-arbitrage conditions on returns to labour and returns to investment. Wages are distributed in proportion to the physical quantity of labour contributed and profits are distributed in proportion to the value of the capital employed in each sector.

Definition 2.6. A *Sraffian price equation* with wages paid *ex ante* is

$$\mathbf{p} = (\mathbf{p}\mathbf{A} + \mathbf{l}w)(1 + r), \tag{2}$$

where $\mathbf{p} = [p_i]$ is a $1 \times n$ row vector of prices. Each p_i is measured in money units per unit of commodity-type i. $w \geq 0$ is the wage rate, measured in money units per unit of labour-time. The 'rate of profit', $r \geq 0$, is a dimensionless ratio of money amounts that scales input costs.

Price equation (2) defines profit-equalising prices of production. For each activity the total price of commodity inputs plus the wage cost multiplied by the uniform rate of profit equals the price of the product. That is, commodity prices are such to cover input costs, profits and wages.

The solution of these kinds of price and quantity equations has been extensively discussed in the literature (e.g., consult [56], [36], [5]).

2.1. **Positive prices.** Assume $\mathbf{I} - \mathbf{A}$ is of full rank, i.e. no column (or row) is a linear combination of the others. Pasinetti [56] argues that this assumption may be justified on practical grounds: '... coefficients are *given* by technology. In general, then, we cannot expect to find among the first $(n-1)$ columns one that just happens to be linearly dependent on the others ... If that were to occur, it would be a truly exceptional coincidence indeed' ([56], p.56). But the real motivation is to guarantee that the Sraffian system contains sufficient independent equations to determine solutions.

Consider $\mathbf{A}$ as a production graph. Each node is an activity. For every $a_{i,j} > 0$ there is an edge that connects activity i to j. Assume that the graph is connected, that is for every i and j there is a route that traverses the edges from i to j. The assumption of connectedness ensures that the economic system forms a connected whole, rather than separate 'islands' of economic activity.

Definition 2.7. A technique is *admissible* if connected, productive and $\text{rank}(\mathbf{I}-\mathbf{A}) = n$.

The analysis is restricted to admissible techniques that produce outputs larger than inputs. But the productivity of a technique is finite. This bound on productivity is captured by the concept of the maximum rate of real growth. It is a function of the dominant root of the technique.

Lemma 2.1. If $\mathbf{A}$ is productive then the dominant root, or eigenvalue of greatest modulus, λ_m, of $\mathbf{A}$ is non-negative and less than unity.

Proof. The eigenvalue equation, $\mathbf{xA} = \lambda \mathbf{x}$ has a dominant root, $\lambda_m \geq 0$, by the Perron-Frobenius theorems for reducible, non-negative square matrices (e.g., see [12], or mathematical appendices of [39], [56] or [36]). $\mathbf{A}$ is productive so $\lambda_m < 1$. □

Lemma 2.2. The maximum rate of real growth, $G > 0$, is $G = (1 - \lambda_m)/\lambda_m$.

Proof. G is defined by replacing the net product of equation (1) with a scalar multiplier, $\mathbf{q} = \mathbf{qA}^{\mathrm{T}}(1+G)$. Rearrange in the form of an eigenvalue equation, $\mathbf{qA}^{\mathrm{T}} = \lambda \mathbf{q}$. By lemma 2.1, $G = (1 - \lambda_m)/\lambda_m$, where $0 \leq \lambda_m < 1$ is the dominant root. □

The maximum rate of real growth is identical to the maximum rate of profit.

Lemma 2.3. The maximum rate of profit, $R > 0$, is $R = G$.

Proof. Profits are maximum, $r = R$, when $w = 0$ such that the value of the net product is entirely appropriated in the form of profits, $\mathbf{p} = \mathbf{pA}(1 + R)$. Rearrange in the form of an eigenvalue equation, $\mathbf{pA} = \lambda \mathbf{p}$. The eigenvalues of matrix are identical to the eigenvalues of the transposed matrix. So by lemma 2.2, $R = G$. □

The following theorem is used to prove the existence and non-negativeness of various inverse matrices that appear in the price and quantity solutions.

Theorem 2.4. For any $n \times n$ matrix $\mathbf{B}$, with $b_{i,j} \leq 0$ for all $i \neq j$, there exists a column vector $\mathbf{x} \in \mathbb{R}^n_+$, such that $\mathbf{Bx} > \mathbf{0}$ if and only if $\mathbf{B}^{-1} \geq \mathbf{0}$.

Proof. A standard result (e.g. by theorem 6.3 of ref [30], p. 232; or p. 134 of ref [4]). (It also follows that $\mathbf{B}$ is an M matrix that satisfies the Hawkins-Simon condition.) □

Lemma 2.5. Sraffian prices are positive and given by

$$\mathbf{p}(w, r) = \mathbf{l}[\mathbf{I} - \mathbf{A}(1 + r)]^{-1} w(1 + r), \tag{3}$$

where $0 \leq r < R$. This equation has two unknowns that by convention are fixed by specifying a normalization condition, the *numéraire*, and a price, either the wage or profit rate.

Proof. Let $\mathbf{B} = \mathbf{I} - \mathbf{A}(1 + r) = [b_{i,j}]$; then $b_{i,j} \leq 0$ for all $i \neq j$. By lemma 2.3 $\mathbf{pA} = \mathbf{p}/(1 + R)$. If $r < R$ then $\mathbf{p}/(1 + r) > \mathbf{p}/(1 + R)$. Hence, $\mathbf{p} > \mathbf{pA}(1 + r)$; therefore matrix $\mathbf{A}(1 + r)$ is productive and there is a column vector $\mathbf{x} \in \mathbb{R}^n_+$ such that $\mathbf{Bx} > \mathbf{0}$. Then by theorem 2.4, $\mathbf{B}^{-1} \geq \mathbf{0}$.

Expand $\mathbf{B}^{-1}$ as a power series

$$\mathbf{B}^{-1} = \sum_{i=0}^{\infty} \mathbf{A}^i (1 + r)^i,$$

where the first term of the series, $\mathbf{A}^0 = \mathbf{I}$. So $\mathbf{B}^{-1}$ cannot have a zero column. Since $\mathbf{l} > \mathbf{0}$ then $\mathbf{x} = \mathbf{l}\mathbf{B}^{-1} > \mathbf{0}$. By lemma 2.3, there is only one $r = R$ when $w = 0$, otherwise by definition $w > 0$. Then $0 \leq r < R$ implies $w > 0$. So $\alpha = w(1+r) > 0$. Prices are $\mathbf{p}(w, r) = \alpha\mathbf{x}$, a product of a positive scalar and a positive vector; hence $\mathbf{p} > \mathbf{0}$.

Due to the rank condition the system contains n equations in $n + 2$ unknowns so has two undetermined variables. □

Definition 2.8. A numéraire *equation*,

$$\mathbf{pb}^{\mathrm{T}} = \nu,$$

sets the absolute scale of the price system, where $\mathbf{b} \geq \mathbf{0}$ is a $1 \times n$ commodity bundle and $\nu > 0$. (For example, $\mathbf{b} = [\ 1 \quad 0 \quad \ldots \quad 0\ ]$ and $\nu = 1$ sets the *numéraire* to $p_1 = 1$).

2.2. **Positive quantities.** Different kinds of commodity-types need to be distinguished in order to specify the precise conditions for solutions with positive quantities. The Leontief inverse contains the information needed to classify commodities.

Definition 2.9. The *Leontief inverse* is

$$\mathbf{L} = \sum_{i=0}^{\infty} \mathbf{A}^i = (\mathbf{I} - \mathbf{A})^{-1} = [\alpha_{i,j}],$$

where $\alpha_{i,j}$ represents the physical quantity of the ith commodity needed in the economic system as a whole in order to eventually obtain the availability of 1 unit of the jth commodity as a net commodity. Thus the $\alpha_{i,j}$ represent the total requirements (direct and indirect requirements) for the production of unit net commodities. $\mathbf{L}$ is a matrix of vertically integrated factor coefficients.

Recall that elements $a_{i,j}$ of matrix $\mathbf{A}$ represent the physical quantity of the ith commodity needed in the jth industry for the production of 1 unit of the jth commodity. These technical coefficients are the *direct* requirements for the production of unit commodities. In contrast, elements $\alpha_{i,j}$ of $\mathbf{L}$ represent the *total direct and indirect* requirements for the production of unit commodities.

Define $\mathbf{X}^{(i)}$ as the ith column and $\mathbf{X}_{(i)}$ as the ith row of $\mathbf{X}$. Then column vector $\mathbf{L}^{(i)}$ represents the total amounts of heterogeneous commodities $1, \ldots, n$ needed to produce 1 net unit of i; and row vector $\mathbf{L}_{(i)}$ represents the total amounts of commodity i needed to produce $\mathbf{1}$ net unit of heterogeneous commodities $1, \ldots, n$.

The meaning of the Leontief inverse can be better understood by analysing its matrix power-series representation, $\mathbf{L} = \sum_{i=0}^{\infty} \mathbf{A}^i$. Assume a 'final demand' such that the net product of the economy consists of 1 unit of each commodity $1, \ldots, n$. The amount of commodity i directly required to generate 1 unit of j as surplus is trivially $\delta_{i,j}$, where $\delta_{i,j} = 1$ if $i = j$ and $\delta_{i,j} = 0$ otherwise. The unit 'final demands' can therefore be represented by the $n \times n$ identity matrix $\mathbf{I}$. The ith column of $\mathbf{A}$ represents the commodity-inputs directly required to produce 1 unit of i. So to produce a unit net product $\mathbf{I}$ requires commodity inputs $\mathbf{IA} = \mathbf{A}$. These inputs require the availability of indirect inputs $\mathbf{A}^2$, which, in turn, require indirect inputs $\mathbf{A}^3$, and so forth, *ad infinitum*. The vertically integrated factor coefficients are then the sum of the power series $\mathbf{I} + \mathbf{A} + \mathbf{A}^2 + \cdots + \mathbf{A}^n$ as $n \to \infty$.

Lemma 2.6. The Leontief inverse, $\mathbf{L} = (\mathbf{I} - \mathbf{A})^{-1}$, exists and $\mathbf{L} \geq \mathbf{0}$, if $\mathbf{A}$ is productive.

Proof. Let $\mathbf{B} = \mathbf{I} - \mathbf{A} = [b_{i,j}]$; then $b_{i,j} \leq 0$ for all $i \neq j$. $\mathbf{A}$ is productive so there is a column vector $\mathbf{x} \in \mathbb{R}^n_+$ such that $(\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{Bx} > \mathbf{0}$. By theorem 2.4, $\mathbf{B}$ is nonsingular and $\mathbf{B}^{-1} \geq \mathbf{0}$ □

Definition 2.10. A *basic* commodity is a commodity required, directly or indirectly, for the production of all commodities.

Assume there is at least one basic commodity.

Lemma 2.7. Commodity i is basic if $\mathbf{L}_{(i)} > \mathbf{0}$.

Proof. $\mathbf{L}_{(i)}$ is the row vector that represents the total quantities of the ith commodity required to produce commodities $j = 1, \ldots, n$ respectively. If every element of $\mathbf{L}_{(i)}$ is greater than zero then commodity i is required for the production of every other commodity. □

Definition 2.11. A *non-basic* commodity is a commodity that is not required, directly or indirectly, for the production of all commodities.

Lemma 2.8. Commodity i is non-basic if an element of $\mathbf{L}_{(i)}$ is zero.

Proof. This immediately follows from lemma 2.7. □

Define $\mathbf{x}[\alpha]$ as the vector formed by removing elements numbered α from $\mathbf{x}$, where α is a sequence of integers. For example, $\mathbf{x}[i]$ is the vector formed by removing the ith element from $\mathbf{x}$. Define $\mathbf{A}[\alpha_1|\alpha_2]$ as the matrix formed by removing rows numbered α_1 and columns numbered α_2 from $\mathbf{A}$, where α_1 and α_2 are sequences of integers. $\mathrm{S}^{\complement}$ is the increasing sequence of indices formed from the complement of the set S.

Definition 2.12. The set of activities S is a *sub-technique* if (i) every activity $i \in \mathrm{S}$ requires, directly or indirectly, the same set R of input activities ($\mathbf{L}^{(i)} = \mathbf{L}^{(j)}$ for all $i, j \in \mathrm{S}$), (ii) every activity $i \in \mathrm{S}$ is required, directly or indirectly, by the same set T of dependent activities ($\mathbf{L}_{(i)} = \mathbf{L}_{(j)}$ for all $i, j \in \mathrm{S}$), and (iii) every pair of activities $i, j \in \mathrm{S}$, require each other, directly, or indirectly, as inputs ($\mathbf{L}_{(i)}[\mathrm{S}^{\complement}] > \mathbf{0}$ for all $i \in \mathrm{S}$). $\mathbf{A}[\mathrm{S}^{\complement}|\mathrm{S}^{\complement}]$ is the inter-activity matrix associated with sub-technique S.

Sub-techniques are connected so that if any activity in the sub-technique is activated than so are all the others. A sub-technique that consists of a single activity is a degenerate subsystem. The set of all basic commodities B forms the *basic* sub-technique.

Definition 2.13. An *ultimate* sub-technique U is a set of activities that are are not required, directly or indirectly, for the operation of any activities not in U; that is $\mathbf{L}_{(i)}[\mathrm{U}] = \mathbf{0}$.

There may be multiple ultimate sub-techniques.

Figure 2 depicts a production graph with different kinds of commodities.

The ultimate sub-techniques are important because if at least one activity is operated in every ultimate sub-technique then all other activities are necessarily operated. If the net product does not meet this condition then the technique contains redundant information.

Definition 2.14. An *admissible net product* has a surplus in at least one commodity from every ultimate sub-technique.

Assume the net product is admissible.

Lemma 2.9. Sraffian quantities are positive and given by

$$\mathbf{q}(\mathbf{n}) = \mathbf{n}(\mathbf{I} - \mathbf{A}^{\mathrm{T}})^{-1}, \tag{4}$$

where $\mathbf{n}$ is an admissible net product.

Proof. Let $\mathbf{B} = \mathbf{I} - \mathbf{A}^{\mathrm{T}} = [b_{i,j}]$; then $b_{i,j} \leq 0$ for all $i \neq j$. $\mathbf{A}^{\mathrm{T}}$ is productive because $\mathbf{A}$ is productive. Then by theorem 2.4, $\mathbf{B}^{-1} \geq \mathbf{0}$.

$\mathbf{B}^{-1} = \mathbf{L}^{\mathrm{T}}$; hence for every ultimate sub-technique, U, by definition, $[\mathbf{B}^{-1}]^{(i)}[\mathrm{U}] = \mathbf{0}$ and $[\mathbf{B}^{-1}]^{(i)}[\mathrm{U}^{\complement}] > \mathbf{0}$ for all $i \in \mathrm{U}$. As $\mathbf{n}$ is an admissible net product, $\mathbf{n}[\mathrm{U}^{\complement}]$ is non-zero for all U, that is there is a final demand for at least one commodity in every ultimate sub-technique. Hence, $\mathbf{n}\mathbf{B}^{-1} > \mathbf{0}$ and quantities are positive.

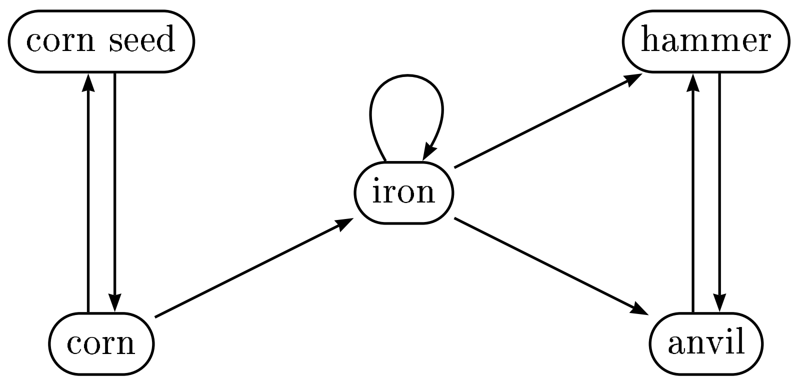


FIGURE 2. **Kinds of commodities**. Corn seed and corn are basic commodities and form the basic sub-technique. Iron is a non-basic and forms a non-basic sub-technique. The hammer and anvil are non-basics and form another non-basic sub-technique. The hammer and anvil also form an ultimate sub-technique. In this small example economy if either the hammer or anvil appear in the net product then all activities are operated.

Due to the rank condition the system contains n equations in $2n$ unknowns, hence has one undetermined $1 \times n$ vector parameter. $\square$

3. Critique of Sraffa's system

In Sraffa's system prices are determined from the wage or profit rate and the available methods of production without reference to subjective utility. The theory of equilibrium prices is dependent upon and secondary to the theory of income distribution. Sraffa establishes a basic proposition of classical economics that 'all the variables of a competitive capitalist system are determined from *outside the market* (i.e., they are determined in a socio-historical context) independent of prices' [71]. This property of Sraffa's system was used, with considerable success, to critique naive marginalist theories of profit that explained income distribution in terms of equilibrium price theory. Sraffa's system refutes the marginalist apology that the profit rate is a return to capital as a scare resource, analogous to land, and a measure of its scarcity [21]. The model emphasises the distributional conflict inherent in capitalist societies, which was central to the analyses provided by Ricardo and Marx, but repressed in marginalist theories. Sraffa derives a reciprocal relationship between the wage level and profit rate: the higher the wage, the lower the average rate of profit.

The subtitle of Sraffa's book – 'Prelude to a Critique of Economic Theory' – suggests that his theoretical insights are not primarily intended to directly function as models of economic reality. Sraffa's theoretical assumptions and abstractions are quite strong and in many respects unrealistic. His single and joint production models are perhaps better viewed as exercises in conceptual analysis under certain stringent assumptions – such as deterministic relations between economic variables and the strict satisfaction of non-arbitrage conditions, such as homogeneous wages and uniform profits – rather than operational models of the economy. For instance, Sraffa's demonstration of the logical possibility of switches in the methods of production is a critique of the logical consistency of marginalism, regardless of any empirical data. Complaints of lack of realism, therefore, cannot count as first-order criticisms of Sraffa's work.

3.1. **Interrupted circular flow.** Kurz and Salvadori [38] explain that in the late 1920's Sraffa sought to elaborate an objectivist approach to economic value revolving around the twin concepts of physical real-cost and social surplus within a framework of production as a circular process or flow. In 1928 Wassily Leontief outlined a similar methodological programme in his article 'The Economy as a Circular Flow' [41]. Both economists were influenced by the physiocrat Quesnay who first pictured the economy as a circular process in his *Tableau Economique* (1758) [66]. Sraffa and Leontief initially used homogeneous systems of simultaneous equations, in which final demand is 'closed' [38, 34], yet by the time of their mature publications, 1960 for Sraffa's PCMC and 1951 for Leontief's second edition of *The Structure of the American Economy* [40], both had moved to inhomogeneous systems of equations, in which final demand is 'open'.

Kohli [34] explains how Leontief's collaboration with the US Bureau of Labor Statistics, beginning in 1941, influenced the development of his open model, which 'proved to be more useful for policymakers than Leontief's first formulation' [34]. The Bureau needed to answer questions about the effect of different spending plans at a time when the US government recognized the need for stimulative fiscal policies. So Leontief 'opened' his model to allow final demands to be determined outside the system of inter-activity relationships. The political need to model the consequences of varying the distribution of an economic surplus directly motivated Leontief's theoretical move from a closed to open model. An open model has the useful property that the net product of the economy is yet to be distributed or allocated; it is an *undistributed surplus*. As Leontief notes, 'For many purposes, such, for example, as the evaluation of quantitative implications of alternative policies in respect to allocation of primary resources or, say, various patterns of public works or governmental purchases, it is necessary to treat the national economy as an open rather than as a closed system' ([40], p.205); and 'one can also determine the effect of which an increase or decrease in any given profit or wage rate would have on the prices of all the goods and services' ([40], p.207).

Sraffa's transition from a closed to open model was motivated by the same requirement to model an undistributed surplus. But the transition left him in a temporary theoretical crisis [53].

Chapter 1 of PCMC, 'production for subsistence', examines an economy without a surplus, formally similar to a closed Leontief model, that 'produces just enough to maintain itself' ([75], p. 3). Sraffa swiftly moves to chapter 2, 'production with a surplus', in order to analyse an economy that 'produces more than the minimum necessary for replacement' ([75], p. 6) in which a net product must be distributed to workers and capitalists, the only 'factors of production' with open-ended and contested consumption requirements. For Sraffa, therefore, the production of the surplus is *unnecessary* for the strict reproduction of the economic system because it can be used in various ways, for additional consumption by workers or capitalists, or for new investment etc. Sraffa's surplus is also an undistributed surplus.

The ordering of closed followed by open model in PCMC reflects the actual course of Sraffa's investigations. His early (1927) equations modelled a closed economy, which he interpreted to be without a surplus ([38], p. 76). In 1928 Sraffa turned to the case of the wage consisting of a share of the surplus product. Sraffa's treatment of income distribution was directly influenced by Ricardo [38], who adopted the instrumental perspective of the capitalist state [66] to study the wage-profit trade-off and the effects of taxation on the social order in his 'Principles of Political Economy and Taxation' (1817) [60].

By 1931 Sraffa encountered a contradiction between the concept of an undistributed surplus and the concept of an uninterrupted circular flow, although he did not use such terms. In a note entitled 'Surplus product' (pp. 87–89 of [38]) Sraffa writes, 'If one attempts to take an entirely objective point of view, the very conception of a surplus melts away' [38]. As every activity in a circular flow has a determinate input and an output, 'there can be no product for which there has not been an equivalent cost, and all costs (=expenses) must be necessary to produce it' [38]. Hence, when considering shares of the surplus product, 'every share distributed must be so for a reason, therefore it is necessary: how can there be a surplus left, unless we assume some sort of indeterminacy?' [38]. Sraffa notes that 'this is a great difficulty: the surplus is the object of the inquiry, but as soon as it is explained, a cause is found for it, and it ceases to be a surplus' [38]. In other words, if a surplus is part of the circular flow, then it is also an input, and therefore a *necessary* reproduction cost; but the surplus is *unnecessary* for reproduction, it is an additional output, and therefore cannot be part of the circular flow.

Sraffa resolved the contradiction by pursuing an open model with an undistributed surplus that is nominally allocated by exogenous distributional parameters, the wage and profit rate. Once the distributional parameters are fixed the nominal cost of the consumed surplus is distributed in the form of wages and profits. But the real distribution of commodity types, that is the composition of worker and capitalist consumption, is absent. In consequence, Sraffian quantity and price equations, such as equations (1) and (2), encode a fundamental asymmetry between non-human and human 'factors of production'. Non-human factors have both inputs and outputs specified in the technique; in contrast, the rates at which workers and capitalists consume the commodity-types that constitute the net product are not specified. The surplus is produced, but not yet distributed, an output, but not a full input.

Sraffa notes that the asymmetry of the price equation has the advantage that distribution can be considered primarily as a nominal wage-profit trade-off, avoiding the need to specify the actual composition of consumption ([57], p.11). A nominal income distribution is consistent with a whole set of real income distributions. But the disadvantage is that the circular flow has been interrupted. Leontief offers a 'closed band' metaphor in an attempt to explain this difference: 'cutting a piece out of a closed band results in [the] appearance of two loose ends. So does the cutting of a closed general equilibrium system. The 'final bill of goods' is the loose end which adjoins the cut out stretch on the demand side. The labor inputs and the inputs of all the other services produced by the household, government, and other sectors of the economy which were eliminated mark the other end or rather the beginning of the new open economy. They must be treated now as original, primary inputs. In a closed system, for example, the level of the labor supply would have been directly connected with the level of real income; that is, the quantities of consumers' goods absorbed by the households. Now such direct connection between the two is disregarded' ([40], p.206).

The turn from closed to open models by two highly influential theorists is probably a major reason for the subordinate role of the closed model in modern discussions of linear production theory (e.g., [56, 36, 5]). But closed models do not represent economies that lack a surplus; rather they can represent economies in which a surplus is physically distributed in constant proportions.

In PCMC Sraffa contrasts Quesnay's picture of a circular flow with the marginalism of his day, which is a 'one-way avenue that leads from 'Factors of production' to 'Consumption goods' ([75], p.93). Yet Sraffa's basic price equation is a one-way avenue from production to surplus. Sraffa's adoption of an undistributed surplus at the cost of an interrupted circular flow is the fundamental cause of some important problems in his theoretical framework.

3.2. **Uniform rate of profit and indeterminate prices.** Sraffa notes that 'one effect of the emergence of a surplus' is the 'appearance of a new class of "luxury" products which are not used, whether as instruments of production or as articles of subsistence, in the production of others' ([75], p.7). These are non-basic commodities.

According to Sraffa the basic sub-technique determines the maximum rate of profits of the economic system. Non-basics, in contrast, cannot be a 'limiting factor' on the growth rate of the economy. This claim creates a problem that Sraffa classifies as a 'freak case'. In an appendix to PCMC Sraffa discusses 'self-reproducing non-basics' (SRNBs). A SRNB is a non-basic commodity i that requires itself for production, that is $a_{i,i} > 0$. Sraffa provides the example of 'beans' that for every 100 units sown no more than 110 are reaped. Sraffa writes, in consequence of the duality of the maximum rate of growth and the maximum rate of profits, that 'it is clear that this would not admit of a rate of profits higher than, or indeed, since other means of production must be used as well, as high as, 10%' ([75], p. 90). If the beans are basic there is no problem – the maximum rate of profit for the economy must be less than 10%, that is $r < 0.1$. But if the beans are non-basic then 'complications arise'

([75], p. 90). If the maximum rate of profit, determined by the basic sub-technique, is higher than 10%, say $R > 0.1$, then the price of beans is infinite when $r = 0.1$ or negative when $r > 0.1$.

Sraffa remarks that beans could still be produced and sold at a positive price on condition that 'the producer sold them at a higher price than the one which, in his book-keeping, he attributes to them as means of production' ([75], p.91). 'Cooking the books' might conceivably work for a single producer because the bean producer can use its own bean output as an input, and impute a cost price for the inputs different from the selling price that prevails in the market. But Sraffa did not fully appreciate the generality of the problem in his appendix on 'beans'. The following proposition is a sketch summary of the problem.

Proposition 3.1. R^* is the maximum rate of profit of the basic sub-technique $\mathbf{A}[B^{\complement}|B^{\complement}]$. If $R^* > R$ then some Sraffian prices are negative for rates of profit $R < r < R^*$.

Proof. (Sketch.) By lemma 2.3 $\mathbf{pA} = \mathbf{p}/(1+R)$. If $R < r < R^*$ then $\mathbf{p}/(1+r) < \mathbf{p}/(1+R)$. Hence, $\mathbf{p} < \mathbf{pA}(1+r)$; therefore matrix $\mathbf{A}(1+r)$ is not productive. The conditions of the proof of lemma 2.5 are not met; hence the positivity of Sraffian price equation (3) cannot be guaranteed. □

In general, a problem occurs if the dominant root of a non-basic sub-technique is greater than the dominant root of the basic sub-technique. In this case, the non-basics have either infinite or negative prices at rates of profits higher than the maximum rate of profit of the non-basic sub-technique [56, 36]. A non-basic sub-technique need not include *any* self-reproducing commodities. The producers of these non-basics *must* get their inputs in the market; hence, Sraffa's attempt to restrict the problem to local book-keeping operations fails. The negative or infinite prices are market prices, not imputed prices. In consequence, Sraffian prices are indeterminate over a subrange of the feasible income distribution, $R \leq r \leq R^*$, where R is the maximum rate of profit of the non-basic sub-technique that has the largest dominant root. Sraffa's price theory is then, at best, incomplete. It was precisely to avoid this problem that we earlier defined the maximum rate of profit to be R, rather than R^*, *contra* Sraffa.

The problem of SRNBs has led, over time, to the rejection of the priority of the basic sub-technique for determining the maximum rate of profit in the economy. For example, Pasinetti (1977) ([56], pp. 108–111) tries to exclude SRNBs on grounds of empirical plausibility. But there is no economic reason why the dominant roots of all non-basic sub-techniques should be less than that of the basic sub-technique. Kurz and Salvadori (1995) [36] reject Sraffa's solution and categorise the problem of SRNBs as an example of a 'limit to the long-period method'. Kurz and Salvadori consider various additional assumptions or restrictions that can mitigate the problem (see, in particular, pages 70–73, 82–84, 107–110, 341–348), including the analysis of short-period demand considerations in an economy of agents with perfect foresight, a model quite far from Sraffa's theoretical framework. Abraham-Frois and Berrebi (1997) [1] identify a class of 'blocking goods' that are defined to constrain the maximum rate of profit. Blocking goods are not necessarily basic commodities. Bidard (2004) [5] rejects the idea that the basic sub-technique is necessarily the limiting factor and instead takes the 'whole economy into account' (p. 31), such that all commodities, whether basic or non-basic, are potentially determinants of the maximum rate of profit. Bidard present this solution as a matter of choosing the correct definition of viability but does not consider how a non-basic can constrain the growth of the whole economy, including the basic sub-technique, if it is not an input to the basic sub-technique and hence has no explicit connection to it.

The fact that the surplus is undistributed, and therefore only an output, entails the existence of non-basic commodities. The circular flow is interrupted and the economy fractures into basic and non-basic sub-techniques with different maximum rates of profit that have feed-forward but not feedback input-output relations. The Sraffian price equation imposes the global constraint of a uniform rate of profit on all activities. But only the basic sub-technique can globally constrain the maximum rate of profit because only it is an input to, and has an influence upon, all the others. The problem of SRNBs arises when the non-basic sub-techniques cannot operate at the basic rate of profit. Their operating constraints cannot feedback to constrain the basic sub-technique. To avoid price indeterminacy the global properties of the economy must be taken into account. But neo-Ricardian theorists do not have a good explanation of how a non-basic sub-technique can globally constrain the system if it is not an input to the basic sub-technique. The root cause of the problem, an interrupted circular flow, has not been identified. Sources (basics) and sinks (non-basics) can appear in Sraffian economies solely in virtue of the undistributed surplus.

Sraffa reminds the reader, in his discussion on 'beans', that 'we are all the time concerned merely with the implications of the assumption of a uniform price for all units of a commodity and a uniform rate of profits on all the means of production. In the case under consideration, if the rate of profits were at or above 10% it would be impossible for these conditions to be fulfilled' ([75], p.91). The existence of SRNBs with growth rates lower than the basic sub-technique implies that, in open models, due to the lack of input-ouput relations between non-basics and basics, there is no economic mechanism for the equalisation of the rate of profit. In such cases once a uniform rate of profit is imposed then prices become negative.

3.3. **Subjectivism.** Around 1928, in relation to his early closed models of an economy, Sraffa wrote:

> The significance of the equations is simply this: that if a man fell from the moon on the earth, and noted the amount of things consumed in each factory and the amount produced by each factory during a year, he could deduce at which values the commodities must be sold, if the rate of interest must be uniform and the process of production repeated. In short, the equations show that the conditions of exchange

> are entirely determined by the conditions of production. (Quoted in [28]).

The metaphor of the 'man from the moon' neatly encapsulates the idea that a disinterested observer who counts inputs and outputs has sufficient information to deduce prices of production. But the adoption of an undistributed surplus entails that Sraffa's surplus equations do not live up to this strict objectivism.

In one sense the wage is a *cost* and enters 'the system on the same footing as the fuel for the engines or the feed for the cattle' ([75], p.9). Both corn and hay are necessary inputs to produce units of labour-power and units of horse-power respectively. Therefore wages must be considered as physical inputs to a household sector that outputs labour-power. But in another sense the wage appears to be an *income* and must be treated differently because it can consist of a share in the surplus. For instance, roses for the labourer is an unnecessary luxury, superfluous although desirable [57].

Sraffa deems it natural, therefore, to split the wage into two parts: a first part that represents physical necessities to reproduce the productive capacity of the labourer (cost), and a second part that represents luxuries (income). Sraffa writes:

> We must now take into account the other aspect of wages since, besides the ever-present element of subsistence, they may include a share of the surplus product. In view of this double character of the wage it would be appropriate, when we come to consider the division of the surplus between capitalists and workers, to separate the two component parts of the wage and regard only the 'surplus' part as variable; whereas the goods necessary for the subsistence of the workers would continue to appear, with the fuel, etc., among the means of production ([75], p.9).

The theoretical options are: (i) a fully symmetric approach, which treats labour like any other commodity, and therefore regards the wage as another set of inter-activity coefficients; or an asymmetric approach, which treats labour differently to other commodities, by either (ii) splitting the wage into its necessary and surplus parts, a partially asymmetric approach, or (iii) treating all the wage as a share of the surplus, a fully asymmetric approach. Despite Sraffa's observation that the partially asymmetric approach is the most appropriate for expressing the 'double character' of the wage he nevertheless opts for the fully asymmetric approach.

> We shall, nevertheless, refrain in this book from tampering with the traditional wage concept and shall follow the usual practice of treating the whole of the wage as variable ([75], p.10).

But his reason for doing so is either not made explicit or is an appeal to convention.

Humans, unlike fixed capital, such as machinery or animals, develop new needs. So the commodity inputs to labour do have a 'double character' if we wish to distinguish between necessaries and luxuries. But inter-activity coefficients simply describe factual input-output relations and are neutral with respect to the cause of those relations. For example, a fully symmetric approach to the wage, in which real wages are explicit inputs to a labour household sector, is fully consistent with the existence of a distinction between necessaries and luxuries. The 'double character' of the wage, one part a physiological lower-bound on consumption and one part luxuries, is irrelevant to the construction of an input-output table that describes the physical inputs and outputs that obtain at any particular time. The distinction between wage as cost and wage as income is superfluous at this level of abstraction. Sraffa's asymmetrical treatment of labour is not justified by the 'double character' of the wage. Leontief remarks, in his 1928 essay, that the concept of a surplus 'is best understood if one enquires into the use of this "free" income. The answer is: it either accumulates or is used up unproductively'. But then adds that 'any attempt to distinguish an unproductive and a productive side to personal consumption is just as arbitrary as it is when dealing with the "objective" cost of goods'.

Labour is a basic commodity. This means that all its inputs are also basic. But the goods consumed by workers are now potentially non-basic because the wage is variable. That's because, as surplus, they are modelled simply as outputs, never as inputs. This is awkward, and Sraffa recognizes this: 'The drawback of this course is that it involves relegating the necessaries of consumption to the limbo of non-basic products' ([75], p. 10). But he assures us that his subsequent discussion 'can easily be adapted to the more appropriate, if unconventional, interpretation of the wage' ([75], p.10) as consisting of two components: one subsistence, the other surplus. Let's check this.

Following Pasinetti [56], the technique augmented by the *subsistence* requirements of the workers is

$$\mathbf{A}^{+} = \mathbf{A} + \mathbf{w}^{\mathrm{T}}\mathbf{l},$$

where $\mathbf{w} \geq \mathbf{0}$ is a $1 \times n$ vector of subsistence consumption coefficients per unit of labour. The Sraffian price equation, with wages paid *post factum*, is then

$$\mathbf{p} = \mathbf{p}\mathbf{A}^{+}(1+r) + \mathbf{l}w,$$

where workers augment their subsistence goods by participating in the consumption of the surplus. Sraffa maintains that non-basic goods have a 'purely passive' ([75], p.7) role and that 'the chief economic implication of the distinction [is] that basics have an essential part in the determination of prices and the rate of profits, while non-basics have none' ([75], p.54). Yet techniques $\mathbf{A}$ and $\mathbf{A}^{+}$ have entirely different sets of basic and non-basic goods conditional on the subsistence vector $\mathbf{w}$. For example, if a non-basic good in technique $\mathbf{A}$ appears in the subsistence vector $\mathbf{w}$ then at least that good is basic in the augmented technique $\mathbf{A}^{+}$. Hence, according to Sraffa, the maximum rate of profit that can prevail in the economy is contingent on the classification of worker's consumption goods.

The classification is not a technical datum, an objective fact, like counting the inputs and outputs at the factory gates. Some kind of objective or causal justification is required to judge whether or not a commodity is necessary for the reproduction of workers. But Sraffa does not provide a principle for classification, nor is it clear how one could be formulated. Hence, absent such a principle, the *subjective* classification decision of the theorist determines the maximum rate of profit of the basic subsystem and hence the whole economy. The man from the moon cannot just count inputs and outputs after all; he must also introduce a moral element in order to classify wage goods.

3.4. **Non-equilibrium.** The concept of equilibrium employed in PCMC is either ambiguous or absent. In chapter 1, devoted to the analysis of a subsistence economy, 'which produces just enough to maintain itself' ([75], p.3), Sraffa writes, 'There is a unique set of exchange-values which if adopted by the market restores the original distribution of the products and makes it possible for the process to be repeated; such values spring directly from the methods of production' ([75], p.3). In chapter 2, devoted to the analysis of a surplus economy, Sraffa provides an example economy in which the means of production used-up in the production period are replaced ([75], p.7) and makes reference to the 'national income of a system in a self-replacing state' ([75], p.11). It seems natural to conclude, particularly given the static nature of simultaneous equations, that Sraffa, throughout PCMC, is analysing an economy in a state of self-replacing equilibrium. For example, Pasinetti begins his classic exposition of Sraffa's single production system with:

> The methods of production are such that each industry produces a single commodity by using a certain physical quantity of labor and certain physical quantities of commodities. These commodities (required as means of production) are completely used up in each period, so they have to be replaced entirely. At the end of each "year" the total output of the system must therefore be divided into two parts: one part must be devoted to the replacement of those commodities which have been used up in the production process; the remainder represents the final net output, or net national income, or net product, and can be devoted to consumption (the system being in a stationary state) ([56], pp.71–72).

But as Ravagnani [59] notes, although this reproduction interpretation is widespread, 'Sraffa never introduces in his analysis any specific assumption about the allocation of the physical surplus' and reminds us that Sraffa, in a comment on Harrod's review of PCMC [76], employs an example with a negative net output of a commodity in which the price equations continue to determine prices. Ravagnani argues that Sraffa's approach is not restricted to a self-replacing equilibrium but has more general applicability. All that Sraffa requires is that the technique be productive. Similarly, Roncaglia claims that 'there is no reference, implicit or explicit, in any of Sraffa's assumptions, to an aggregate equality between the commodities produced and supplied on the one hand, and the quantities demanded and sold, on the other' ([64], p. xvii).

The reason for such ambiguity and contrasting interpretations is simple. A circular flow interpretation of Sraffa's theoretical framework implies that the object of inquiry is an economy in a state of self-replacing equilibrium. But Sraffa's surplus is undistributed, the allocation of the physical surplus is absent, and hence the flow interrupted. An interrupted circular flow may be continued in a number of ways, self-replacement just being one of many possibilities. Sraffa does is not clear whether the production of the surplus is a process that repeats (an uninterrupted circular flow with a distributed surplus) or whether it is a singular event (an interrupted circular flow with an undistributed surplus).

3.5. **Proposal.** Sraffa, in PCMC, presents a circular flow approach to economic analysis, influenced by Quesnay's *Tableau Economique* and Petty's objectivism, implicitly rejecting Marshall's partial equilibrium approach and the marginalism of his time [38]. Influenced by Ricardo's treatment of the problem, Sraffa also analyses variations in the distribution of income. The circular flow metaphor implies a process of self-replacing equilibrium, whereas the income distribution metaphor implies a distributional event. These two basic metaphors clash. Either the surplus is distributed and the flow is circular, or the surplus is undistributed and the flow is interrupted. As Joan Robinson noted in her review of PCMC [62], 'we are concerned with equilibrium prices and a rate of profit uniform throughout the economy, but we are given only half an equilibrium system to stand on'. Sraffa has one foot in and one foot out of the circular flow.

I therefore propose an analytical experiment: close Sraffa's system, assume simple reproduction with a specified composition of net output, and repair the circular flow. By performing this experiment we will discover that Sraffa's interrupted circular flow has a further unfortunate consequence: it hides the value-theoretic principle of objective real-cost.

4. Closing Sraffa

A circular flow in a state of self-replacing equilibrium can be represented by a closed system of equations in which the surplus is fully distributed. The Sraffian equations are closed by adding explicit worker and capitalist household sectors. By combining the information contained in Sraffa's price and quantity equations we derive the inputs and outputs of the worker and capitalist households.

Definition 4.1. The *real wage*, or worker consumption, is a $1 \times n$ row vector, $\mathbf{w} = [w_i]$. Each $w_i \geq 0$ represents the physical quantity of commodity-type i consumed by workers. Workers consume at least one commodity so there is a j such that $w_j > 0$.

Definition 4.2. *Capitalist consumption* is a $1 \times n$ row vector, $\mathbf{c} = [c_i]$. Each $c_i \geq 0$ represents the physical quantity of commodity-type i consumed by capitalists. Capitalists consume at least one commodity so there is a j such that $c_j > 0$.

Definition 4.3. A *Sraffian system* Ψ is a price equation,

$$\mathbf{p} = (\mathbf{p}\mathbf{A} + \mathbf{l}w)(1 + r), \tag{5}$$

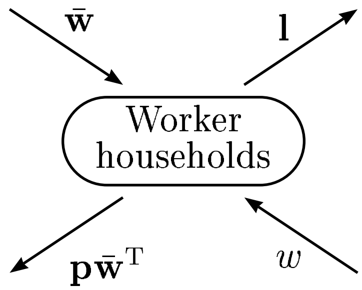


FIGURE 3. **Inputs and outputs of worker household sector. Inputs**: the real-wage $\bar{\mathbf{w}}$ and the nominal wage-rate w per unit of labour. **Outputs**: the wage bill $\mathbf{p}\bar{\mathbf{w}}^{\mathrm{T}}$ per unit of labour, and direct labour $\mathbf{l}$ performed per unit of commodity produced.

a quantity equation,

$$\mathbf{q} = \mathbf{q}\mathbf{A}^{\mathrm{T}} + \mathbf{w} + \mathbf{c}, \tag{6}$$

and a 4-tuple $(\mathbf{A}, \mathbf{l}, \mathbf{w}, \mathbf{c})$, where $\mathbf{A}$ and $\mathbf{n} = \mathbf{w} + \mathbf{c}$ are admissible. The technique, direct labour coefficients, real wage, and capitalist consumption are independent variables.

The Sraffian system represents sufficient information to construct a circular flow representation of simple reproduction. Other starting points, which also embody sufficient information, in terms of different independent variables, are possible. The independent variables of the Sraffian system are quantities that in principle can be measured.

4.1. **Worker households.** The outputs of the worker household sector are units of labour. The inputs to the worker sector are a bundle of consumption commodities.

The labour outputs per unit of commodity produced, $\mathbf{l}$, are known. We need to deduce the commodity inputs per unit of labour produced, that is consumption rates per unit labour.

Lemma 4.1. Ψ determines the composition of gross output, $\mathbf{q}$.

Proof. Ψ specifies the real wage and capitalist consumption, hence gross output is determined by equation (4). □

Definition 4.4. The total *quantity of labour*, L, is the amount of labour directly applied during the production period.

Lemma 4.2. Ψ determines the total quantity of labour, $L = \mathbf{q}\mathbf{l}^{\mathrm{T}}$.

Proof. The total quantity of direct labour is determined by gross output and the direct labour coefficients. By lemma 4.1 gross output is determined by the Sraffian system. □

Definition 4.5. The *worker consumption coefficients* are a $1 \times n$ row vector, $\bar{\mathbf{w}} = [\bar{w}_i]$. Each $\bar{w}_i \geq 0$ represents the physical quantity of commodity-type i directly required to output 1 unit of labour-power.

Lemma 4.3. Ψ determines worker consumption coefficients, $\bar{\mathbf{w}} = \mathbf{w}/L$.

Proof. Worker consumption coefficients are determined by the real wage and the total quantity of labour. By lemma 4.2 the total quantity of labour is determined by the Sraffian system. $\square$

Figure 3 depicts the inputs and outputs of worker households.

4.2. **Capitalist households.** The output of the capitalist household sector is money to fund production. The inputs to the capitalist sector are a bundle of consumption commodities.

Specifying the output of the capitalist sector is a subtle task for two reasons. First, we need an account of the property relations between capitalist households and firms in order to model the supply of working capital and the receipt of profit income. Second, the output of the capitalist sector is money and hence, unlike any other sector, the output coefficients have a direct relationship to commodity prices.

The firm is the legal party that owns all produced outputs, all the liabilities for used-up inputs and all the discretionary control rights over the work process [15]. In contrast, legal parties that supply only inputs, such as employees or input suppliers, do not possess such property rights. Under capitalism the suppliers of equity capital (stockholders) assume the ownership rights of the firm and thereby claim a residual profit-income from firm revenue, once capital and labour costs are paid. Hence, money, in the hands of capitalists, can function as money-capital because its supply to production commands a return. In constrast, money, in the hands of consumers, that is either workers or capitalists spending for personal consumption, does not command a return and functions only as means of exchange. Although consumers do not charge producers for the service of buying goods, capitalists do charge firms for the service of supplying capital. Money-capital is simply money that performs a different function in virtue of capitalist property rights: when spent, it increases in value and returns (under conditions of self-replacing equilibrium).

The supply of money-capital to fund production is an objective and socially necessary feature of capitalist production. Hence, under conditions of capitalist production, money-capital is a 'factor of production' like any other commodity input. Input-output coefficents describe the transmission of heterogenous materials and services between productive activities. At this level of abstraction the productive activities can be viewed as capitalist firms. The role of capitalist as the residual claimant in the firm can be represented in an input-output model by treating money-capital as a commodity with a price.

Definition 4.6. The *direct money-capital coefficients* are a $1 \times n$ row vector, $\bar{\mathbf{m}} = [\bar{m}_i]$. Each $\bar{m}_i > 0$ represents the nominal quantity of money-capital directly supplied to output 1 unit of commodity-type i. $\bar{\mathbf{m}}$ represents unit input costs for each commodity type, where costs include the cost of commodity inputs and wages.

Money-capital coefficients are implicitly defined by a Sraffian system Ψ.

Lemma 4.4. Ψ determines the rate of profit r.

Proof. By lemma 4.3 worker consumption coefficients are determined by the Sraffian system. The wage rate, w, exactly covers the cost of the real wage, $\mathbf{w}$; that is, $w = \mathbf{p}\bar{\mathbf{w}}^{\mathrm{T}}$. Substituting for w in equation (5) gives

$$\mathbf{p} = (\mathbf{pA} + \mathbf{p}\bar{\mathbf{w}}^{\mathrm{T}}\mathbf{l})(1+r) \tag{7}$$
$$= \mathbf{pA}^{+}(1+r). \tag{8}$$

Let $\lambda_* = 1/(1+r)$ and rearrange to get

$$\mathbf{pA}^{+} = \lambda_* \mathbf{p}. \tag{9}$$

The maximum eigenvalue solution of equation (9) yields λ_* and hence the profit rate $r = (1/\lambda_*) - 1$. ☐

Prices and the wage rate are determined up to the choice of *numéraire*. Once the scale of the price system is fixed the money-capital coefficients are determined.

Lemma 4.5. Ψ determines the money-capital coefficients, $\bar{\mathbf{m}} = \mathbf{p}/(1+r)$, where $\mathbf{p}$ is the left-hand eigenvector of $\mathbf{A}^{+}$ associated with the maximum eigenvalue.

Proof. Ψ determines r by lemma 4.4. $\mathbf{p}$ is the left-hand eigenvector of $\mathbf{A}^{+}$ associated with $\lambda_* = 1/(1+r)$. By lemma 2.5, prices are positive with two unknowns. Specifying the real wage eliminates w; hence $\mathbf{p}$ has one unknown, which is fixed by a *numéraire* equation.

The money-capital coefficients, $\bar{\mathbf{m}}$, represent the nominal amounts of working capital required by each activity during the production period in order to pay for input materials, including labour. Equivalently, the coefficients represent the unit costs for each sector that are marked-up by the prevailing rate of profit to determine output prices. The money-capital coefficients are unique, therefore, in having the following direct relationship to commodity prices,

$$\mathbf{p} = \bar{\mathbf{m}} + \bar{\mathbf{m}}r = \bar{\mathbf{m}}(1+r). \tag{10}$$

Hence,

$$\bar{\mathbf{m}} = \frac{1}{1+r}\mathbf{p}, \tag{11}$$

which is a scalar multiple of prices, as required. ☐

Ψ determines the supply of money-capital from the household sector to the n production activities. Let's consider the monetary exchanges between the capitalist sector and a single industry i.

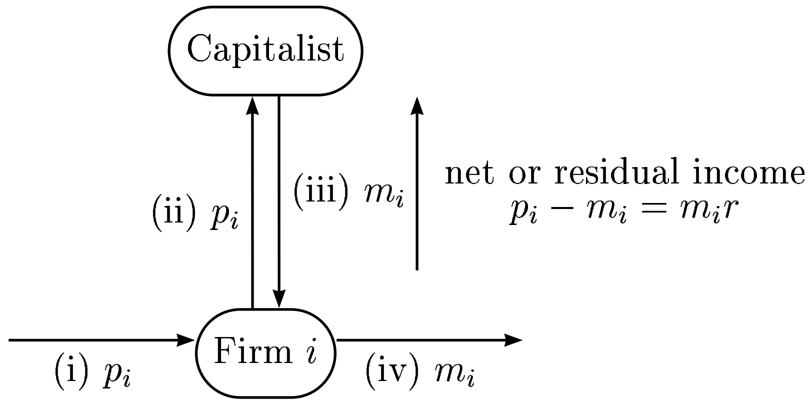


FIGURE 4. **Monetary exchange between capitalists and firms.** (i) Firm i sells unit output and receives p_i revenue. (ii) The revenue is transferred to the capitalist owner(s). (iii) The capitalist supplies m_i money to the firm to fund the production period. This money functions as money-capital and receives a return. (iv) The firm spends its working capital to cover the input costs incurred during the production period. The net monetary exchange between capitalist and firm is the residual profit income, $m_i r$, where r is the rate of profit.

To produce 1 unit of commodity i requires the supply of m_i units of money-capital to pay for used-up inputs and wages. Once supplied, the money-capital reverts to means of exchange, and is transferred to input suppliers, which are other industries and worker households. Industry i sells the unit product on the market and receives p_i revenue. The revenue is transferred to the capitalist owners. A part of the revenue, m_i, is allocated as money-capital to fund the next 'round' of production. Hence, in simple reproduction, the supply of money-capital is not an injection of additional money into circulation but the allocation of firm revenue by capitalist owners to meet the liabilities incurred during production. Once input costs are met the residual income is $p_i - m_i$. By equation (10) $p_i - m_i = m_i r$ and so profit is $m_i r$.

Figure 4 depicts the monetary exchanges between capitalists and firms. The capitalist sector supplies m_i units of money-capital and receives *gross* revenue p_i, part of which replaces the used-up money-capital, leaving a *net*, or *residual*, profit-income $m_i r$. This profit income may then be spent on capitalist consumption goods (in simple reproduction capitalists neither save nor make new investments). The output of the capitalist households, money-capital m_i, therefore commands the price, r, which is identically the rate of profit, a ratio measured in units of currency per unit of currency. 'It is not until capital is money-capital that it becomes a commodity, whose capacity for self-expansion has a definite price quoted every time in every prevailing rate of interest' [45].

Note that this model of profit income is not an addition to Sraffa's theoretical framework but is implicit within it. Once the Sraffian system is defined the deduction of the money-capital coefficients follows as a logical necessity.

We can now determine the final item of information required to close Sraffa's open representation: the capitalist household consumption rates per unit of money-capital supplied.

Definition 4.7. The total *quantity of money-capital supplied*, or *total money-capital*, M, is the nominal amount of money-capital supplied in the production period.

Lemma 4.6. Ψ determines the total money-capital, $M = \mathbf{q}\bar{\mathbf{m}}^{\mathrm{T}}$.

Proof. The quantity of money-capital supplied is determined by gross output and the money-capital coefficients. By lemmas 4.1 and 4.5 both are determined by the Sraffian system. □

Definition 4.8. The *capitalist consumption coefficients* are a $1 \times n$ row vector, $\bar{\mathbf{c}} = [\bar{c}_i]$. Each $c_i \geq 0$ represents the physical quantity of commodity-type i directly required to output 1 unit of money-capital.

Lemma 4.7. Ψ determines capitalist consumption coefficients, $\bar{\mathbf{c}} = \mathbf{c}/M$.

Proof. By lemma 4.6 capitalist consumption coefficients are determined by the Sraffian system. □

Capitalists receive $1r$ units of money for every unit of money-capital supplied to production. The capitalist class spend the profit-income on consumption.

Lemma 4.8. Given Ψ, the rate of profit is the total price of capitalist consumption per unit of money-capital supplied; that is

$$r = \mathbf{p}\bar{\mathbf{c}}^{\mathrm{T}}. \tag{12}$$

Proof. See appendix A. □

The equality of the price of money-capital and the cost of capitalist consumption, $r = \mathbf{p}\bar{\mathbf{c}}^{\mathrm{T}}$, is the counterpart of the equality of the price of labour and the cost of the real wage, $w = \mathbf{p}\bar{\mathbf{w}}^{\mathrm{T}}$.

Equation (12) is an instance of Kalecki's principle ([33], Ch. 3) that 'capitalists earn what they spend' ([79], Ch. 3). Kalecki deduces a macroeconomic accounting identity in a closed economy between gross profits and the sum of gross investment plus capitalists' consumption. In simple reproduction, gross investment is zero, hence Kalecki's principle implies that gross profits equal capitalists' consumption. Equation (12) implies $\mathbf{p}\mathbf{c}^{\mathrm{T}} = \bar{\mathbf{m}}\mathbf{q}^{\mathrm{T}}r = (r/(1+r))\mathbf{p}\mathbf{q}^{\mathrm{T}}$; that is total capitalist earnings ($\bar{\mathbf{m}}\mathbf{q}^{\mathrm{T}}r$) are equal to total capitalist expenditures ($\mathbf{p}\mathbf{c}^{\mathrm{T}}$).

Figure 5 depicts the inputs and outputs of capitalist households.

Proposition 4.1. The Sraffian system Ψ determines

(1) worker consumption coefficients, $\bar{\mathbf{w}}$,
(2) direct money-capital coefficients, $\bar{\mathbf{m}}$, and

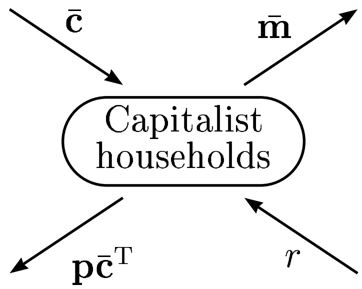


FIGURE 5. **Inputs and outputs of capitalist household sector. Inputs**: capitalist consumption $\bar{\mathbf{c}}$ and the price of money-capital r per unit of money-capital supplied. **Outputs**: the consumption bill $\mathbf{p}\bar{\mathbf{c}}^{\mathrm{T}}$ per unit of money-capital supplied, and direct money-capital coefficients $\bar{\mathbf{m}}$ supplied per unit of commodity produced.

(3) capitalist consumption coefficients, $\bar{\mathbf{c}}$,

which are dependent variables.

Proof. By lemmas 4.3, 4.5 and 4.7 respectively. □

Proposition 4.1 simply states that inputs to the worker sector, $\bar{\mathbf{w}}$, and the inputs and outputs of the capitalist sector, $\bar{\mathbf{c}}$ and $\bar{\mathbf{m}}$, can be deduced from the open Sraffian system Ψ, once the size and composition of the net product is specified.

We now have sufficient information to construct a closed, circular flow representation of Sraffa's equations.

5. Simple reproduction as a monetary-production circular flow

Although the circular flow representation of simple reproduction can be constructed from a Sraffian system it is an independent economic model. So first we specify the model; then we relate it to the Sraffian system. The main feature of the circular flow is that the surplus is fully distributed, the flow is uninterrupted, and money-capital is a commodity. But we shall see that it is nonetheless formally equivalent to a Sraffian system. It is a different view of the same economic reality. But the new viewpoint discloses properties that are otherwise hidden by Sraffa's surplus representation.

Definition 5.1. The circular flow representation of simple reproduction, or *circular flow*, is a $n+2 \times n+2$ matrix,

$$\mathbf{C} = \left[\begin{array}{ccc} \mathbf{A}^{\circ} & [\bar{\mathbf{w}}^{\circ}]^{\mathrm{T}} & [\bar{\mathbf{c}}^{\circ}]^{\mathrm{T}} \\ \mathbf{l}^{\circ} & \alpha & 0 \\ \bar{\mathbf{m}}^{\circ} & 0 & \beta \end{array} \right] = [c_{i,j}],$$

where:

(1) $\mathbf{A}^{\circ} = [a_{i,j}]$ is a $n \times n$ matrix of inter-activity coefficients. Each $a_{i,j} \geq 0$ represents the physical quantity of commodity-type i directly required to output 1 unit of commodity-type j.
(2) $\bar{\mathbf{w}}^{\circ} = [w_i]$ is a $1 \times n$ row vector of worker consumption coefficients. Each $w_i \geq 0$ represents the physical quantity of commodity-type i directly required to output 1 unit of labour-power.
(3) $\bar{\mathbf{c}}^{\circ} = [c_i]$ is a $1 \times n$ row vector of capitalist consumption coefficients. Each $c_i \geq 0$ represents the physical quantity of commodity-type i directly required to output 1 unit of money.
(4) $\mathbf{l}^{\circ} = [l_i]$ is a $1 \times n$ row vector of direct labour coefficients. Each $l_i > 0$ represents the physical quantity of labour-power directly required to output 1 unit of commodity-type i.
(5) $\bar{\mathbf{m}}^{\circ} = [m_i]$ is a $1 \times n$ row vector of direct money-capital coefficients. Each $m_i > 0$ represents the nominal amount of money-capital directly supplied to output 1 unit of commodity-type i.
(6) The scalar $0 \leq \alpha \leq 1$ is the the fraction of the labour time devoted to non-productive activity. $\alpha = 0$ implies the all labour time is devoted to production and $\alpha = 1$ implies all labour time is devoted to leisure.
(7) The scalar $0 \leq \beta \leq 1$ is the capitalists' propensity to save, that is the fraction of capitalist income not spent on consumption. $\beta = 0$ implies zero savings and $\beta = 1$ implies full savings.

Commodity types are labelled $1, 2, \ldots, n+2$. By convention commodity-type $n+1$ is labour-power and commodity-type $n+2$ is money-capital; hence, the first n commodity types are those conventionally considered as technical factors of production. Money is supplied by capitalists at a price, and is therefore a commodity like any other; however, it is not directly produced by labour and may therefore be considered a pure nominal representation, like paper money. The capitalist class enjoys a monopoly in the activity of supplying money-capital to the n production sectors. Money-capital is not directly supplied to the labour sector because it consists of domestic households that rent-out labour, rather than capitalist-owned firms that produce labour for profit.

Figure 6 depicts the circular flow.

Some conditions must be met in order for the circular flow to represent an economy in a state of self-replacing equilibrium.

The money-capital must match unit costs. This condition is specified by the following constraint.

Definition 5.2. The *money-capital constraint* is

$$\bar{\mathbf{m}}^{\circ} = \lambda_m \mathbf{p}' = [m_i] \tag{13}$$

and

$$\bar{\mathbf{m}}^{\circ}[\bar{\mathbf{c}}^{\circ}]^{\mathrm{T}} = (1 - \lambda_m)(1 - \beta), \tag{14}$$

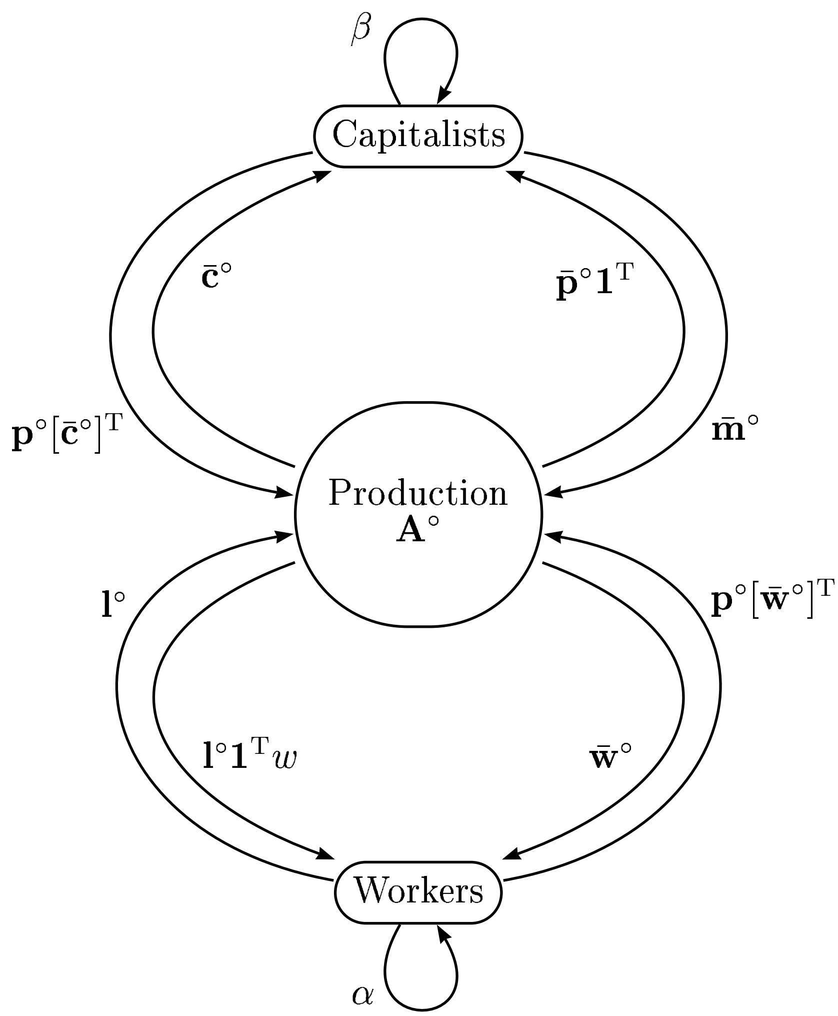


FIGURE 6. **Simple reproduction as circular flow**. $\mathbf{A}^{\circ}$ specifies input-output relations between n activities. Money exchanges are represented by scalars and commodity exchanges by vectors. Capitalists supply money-capital, $\bar{\mathbf{m}}^{\circ}$, to fund production, and receive a gross return $\mathbf{p}^{\circ}\mathbf{1}^{\mathrm{T}}$. A part, $\bar{\mathbf{m}}^{\circ}$, funds the next round of production, and a part, $\bar{\mathbf{m}}^{\circ}\mathbf{1}^{\mathrm{T}}r$, where r is the rate of profit, is net profit-income. A fraction, β, is saved. The remainder $\mathbf{p}^{\circ}[\bar{\mathbf{c}}^{\circ}]^{\mathrm{T}}$ is spent on consumption goods $\bar{\mathbf{c}}^{\circ}$. The working class provides labour $\mathbf{l}^{\circ}$ to production and receives income $\mathbf{l}^{\circ}\mathbf{1}^{\mathrm{T}}w$, where w is the wage rate. The wage is spent as $\mathbf{p}^{\circ}[\bar{\mathbf{w}}^{\circ}]^{\mathrm{T}}$ on consumption goods $\bar{\mathbf{w}}^{\circ}$. A fraction of the available labour time, α, is devoted to non-productive activity.

where $\lambda_m = 1/(1+r)$ is the maximum eigenvalue of $\mathbf{A}^+ = \mathbf{A}^\circ + [\bar{\mathbf{w}}^\circ]^{\mathrm{T}}\mathbf{l}^\circ$, and $\mathbf{p}'$ is the associated left-hand eigenvector of $\mathbf{A}^+$. $\bar{\mathbf{m}}^\circ$ and $\bar{\mathbf{c}}^\circ$ are determined up to the choice of *numéraire*, which is fixed by a *numéraire* equation. Equation (13) ensures that money-capital supplied matches unit costs and equation (14) ensures that capitalist income and expenditure balance.

Money-capital is just another commodity in the circular flow, except for the special property that its quantity is directly related to commodity prices, as expressed by the money-capital constraint. As before, the *net* or *residual* transfer of money from sector i to the capitalist household sector is $m_i r$ (see figure 6). In the case of zero saving, $\beta = 0$, profit income is identical to the price of capitalist consumption. In the case of non-zero savings, $0 < \beta < 1$, a proportion of profit income is deposited in savings accounts and not spent on consumption goods.

Every activity must trade with every other, at least indirectly. This is captured by the concept of irreducibility [26].

Definition 5.3. Let S be a subset of the indices $1, \ldots, n+2$ and let S' be its complement. The circular flow $\mathbf{C}$ is *irreducible* if for all S there is a $c_{i,j} > 0$ for $j \in S$, $i \in S'$. Otherwise, it is *reducible.*

A rank condition, rank $\mathbf{I} - \mathbf{C} = n+1$, ensures that the total volume of demand is equal to the productive capacity of the system [56]. It also ensures there are $n+1$ independent equations for $n+2$ unknowns, and hence solutions are determined up to the choice of *numéraire*. The rank condition implies $\det[\mathbf{I} - \mathbf{C}] = 0$ and hence 1 is an eigenvalue of $\mathbf{C}$ (cf. [56], p. 59).

Definition 5.4. An *admissible* circular flow satisfies the following conditions:

(1) $\mathbf{C} \geq \mathbf{0}$ is a non-negative square matrix.
(2) $\mathbf{C}$ is irreducible.
(3) rank $\mathbf{I} - \mathbf{C} = n+1$.
(4) $\mathbf{C}$ satisfies the money-capital constraint.
(5) $\mathbf{l}^\circ > \mathbf{0}$, i.e. labour is directly required for the production of all n commodities.

Now we show that the Sraffian system implicitly defines a circular flow representation of simple reproduction.

Theorem 5.1. The Sraffian system Ψ uniquely determines an admissible circular flow $\mathbf{C}_\Psi$.

Proof. By proposition 4.1, Ψ determines $\bar{\mathbf{w}}$, $\bar{\mathbf{m}}$ and $\bar{\mathbf{c}}$. Set $\mathbf{A}^\circ = \mathbf{A}$, $\mathbf{l}^\circ = \mathbf{l}$, $\bar{\mathbf{w}}^\circ = \bar{\mathbf{w}}$, $\bar{\mathbf{m}}^\circ = \bar{\mathbf{m}}$, and $\bar{\mathbf{c}}^\circ = \bar{\mathbf{c}}$ to give the circular flow

$$\mathbf{C}_\Psi = \begin{bmatrix} \mathbf{A} & \bar{\mathbf{w}}^{\mathrm{T}} & \bar{\mathbf{c}}^{\mathrm{T}} \\ \mathbf{l} & 0 & 0 \\ \bar{\mathbf{m}} & 0 & 0 \end{bmatrix}.$$

(1) $\mathbf{C}_\Psi \geq \mathbf{0}$ is a non-negative square matrix by definition of Ψ.

(2) Assume $\mathbf{C}_\Psi$ is reducible. Then there is a closed subset of activities that do not need inputs from the remaining activities. Activities $1, \ldots, n$ all require direct inputs of labour and money-capital, due to the positivity of $\mathbf{l}$ and $\bar{\mathbf{m}}$. So any closed subset necessarily contains the worker and capitalist sectors. But by definition, workers and capitalists consume at least one commodity, and the net product is admissible, that is all final commodities are consumed. So the worker and capitalist sectors require, directly or indirectly, inputs from all other activities. Thus there cannot be a closed subset of activities that do not need inputs from the remaining activities; hence $\mathbf{C}$ is irreducible (cf. [6], p. 25).

(3) If rank $\mathbf{I} - \mathbf{C}_\Psi = n + 1$ then any one row of the matrix $\mathbf{I} - \mathbf{C}_\Psi$ is a linear combination of the remaining $n + 1$ rows.

Any row $i \in [1, n + 2]$ is a linear combination of the other rows if there is an $\mathbf{x} = [x_i]$ such that

$$\left[\begin{array}{ccc} -\bar{\mathbf{m}} & 0 & 1 \end{array} \right] = \mathbf{x} \left[\begin{array}{ccc} \mathbf{I} - \mathbf{A} & -\bar{\mathbf{w}}^{\mathrm{T}} & -\bar{\mathbf{c}}^{\mathrm{T}} \\ -\mathbf{l} & 1 & 0 \end{array} \right].$$

Expand to get three equations: (i) $-\bar{\mathbf{m}} = \mathbf{x}[n + 1](\mathbf{I} - \mathbf{A}) - x_{n+1}w$, (ii) $0 = -\mathbf{x}[n+1]\bar{\mathbf{w}}^{\mathrm{T}} + x_{n+1}$, and (iii) $1 = -\mathbf{x}[n+1]\bar{\mathbf{c}}^{\mathrm{T}}$. Set $\mathbf{x} = -(1/r)[\ \mathbf{p} \quad w\ ]$ and substitute into the three equations. Then (i) is $\mathbf{p} = \bar{\mathbf{m}} + \bar{\mathbf{m}}r$ (as $\bar{\mathbf{m}} = \mathbf{pA} + \mathbf{l}w$); (ii) is $w = \mathbf{p}\bar{\mathbf{w}}^{\mathrm{T}}$; and (iii) is $r = \mathbf{p}\bar{\mathbf{c}}^{\mathrm{T}}$. Hence, all three equations are satisfied by the Sraffian system (see lemma 4.5 and lemma 4.8) and so there is an $\mathbf{x}$ that satisfies the condition and rank $\mathbf{I} - \mathbf{C}_\Psi < n + 2$.

Excluding row $n + 2$, any row $i \in [1, n + 1]$ is a linear combination of the remaining rows if there is an $\mathbf{x} = [x_i]$ such that

$$\left[\begin{array}{ccc} -\mathbf{l} & 1 & 0 \end{array} \right] = \mathbf{x} \left[\begin{array}{ccc} \mathbf{I} - \mathbf{A} & -\bar{\mathbf{w}}^{\mathrm{T}} & -\bar{\mathbf{c}}^{\mathrm{T}} \end{array} \right].$$

Expand to get three equations: (i) $-\bar{\mathbf{l}} = \mathbf{x} - \mathbf{xA}$, (ii) $1 = -\mathbf{x}\bar{\mathbf{w}}^{\mathrm{T}}$, and (iii) $0 = -\mathbf{x}\bar{\mathbf{c}}^{\mathrm{T}}$. Consider (iii), $\mathbf{x}\bar{\mathbf{c}}^{\mathrm{T}} = 0$, which implies that $\mathbf{x}$ is orthogonal to the non-zero vector $\bar{\mathbf{c}}^{\mathrm{T}} \geq \mathbf{0}$ in the positive orthant. Hence, there is an $x_i > 0$. Now rearrange (i) to get $\mathbf{x} = -\mathbf{l}(\mathbf{I} - \mathbf{A})^{-1} = -\mathbf{lL}$. $\mathbf{A}$ is productive so $\mathbf{L} \geq \mathbf{0}$ and, by definition, $\mathbf{l} > \mathbf{0}$; therefore, $\mathbf{x} \leq \mathbf{0}$. But this contradicts the requirement that there is an $x_i > 0$. Hence, no $\mathbf{x}$ can satisfy the condition and therefore the sub-matrix excluding row $n + 2$ is of full rank.

Combining, we deduce rank $\mathbf{I} - \mathbf{C}_\Psi = n + 1$.

(4) $\bar{\mathbf{m}}$ satisfies the money-capital constraint by lemmas 4.5 and 4.8.

(5) $\mathbf{l} > \mathbf{0}$ by definition of Ψ.

Hence $\mathbf{C}_\Psi$ is an admissible circular flow. □

This theorem means that a Sraffian system is equivalent to a circular flow with zero savings and full employment of labour time.

Now we will analyse the properties of the circular flow independent of any relations to Sraffa's system. Once this is done, we will show that the properties of the circular flow also hold in Sraffa's system.

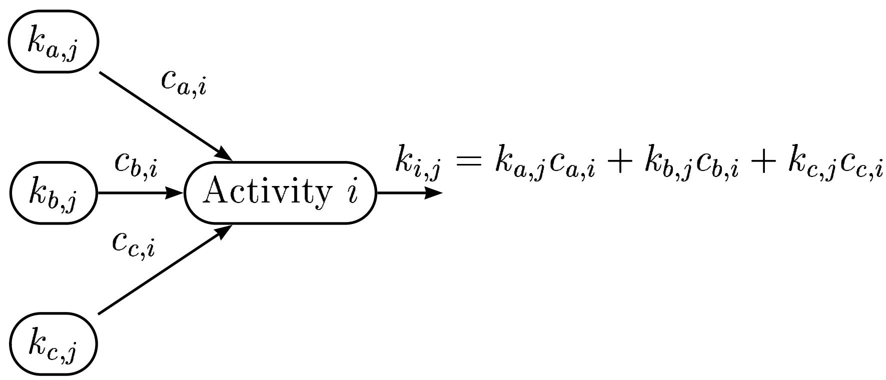


FIGURE 7. **Local real-cost conservation**. The j-cost of commodity i is the weighted sum of the unit j-cost of all its inputs, in this case commodity-types a, b and c.

5.1. **Real-cost accounting in the circular flow.** Linear production price and quantity equations are conservation laws, or accounting identities, that hold in a state of self-replacing equilibrium. The standard approach in linear production theory is to formulate a price equation, which expresses the intra-activity conservation of nominal costs and revenues, and a quantity equation, which expresses the inter-activity conservation of physical inputs and outputs. Here we initially depart from convention by first considering the conservation of real-costs.

To manufacture, say, 1 unit of corn in the circular flow requires various direct inputs; but these direct inputs, in turn, require definite amounts of indirect inputs, and so on. In consequence, a unit of any given commodity requires a definite amount of other commodities to be produced in the economy as a whole.

Definition 5.5. The *real-cost* of commodity i in terms of commodity j, or *j-cost of* i, denoted $k_{i,j}$, is the total amount of j used-up in the economy as a whole during the production of 1 unit of i.

$\mathbf{K} = [k_{i,j}]$ is the $n+2 \times n+2$ matrix of real-costs. Column vector $\mathbf{K}^{(j)}$ represents the real-costs of commodity-types $1, \ldots, n+2$ in terms of j. Row vector $\mathbf{K}_{(j)}$ represents the real-cost of commodity-type j in terms of commodity-types $1, \ldots, n+2$.

In the circular flow all output is productively consumed. So the total amount of commodity-type i used-up in the economy as a whole during the production of 1 unit of i is by definition 1 unit. For example, if $k_{i,i} > 1$ then production of commodity-type i is unproductive and any existing stocks in the economy eventually exhaust. If $k_{i,i} < 1$ then production of commodity-type i is productive and stocks accumulate over time.

Definition 5.6. The real-cost of i in terms of i is 1, that is $k_{i,i} = 1$ for $i = 1, \ldots, n+2$.

Consider accounting for the real-cost of activity $i \in [1, n+2]$. Activity i takes $m \leq n+2$ commodities as direct input and generates a single commodity i as output. For example, assume that activity i only requires commodity-types a, b and c as direct inputs, that is $c_{a,i} > 0$, $c_{b,i} > 0$, $c_{c,i} > 0$, where all other input coefficients are zero. To produce 1 unit of a requires the productive consumption of $k_{a,j}$ units of j in the economy as a whole. But to produce 1 unit of i only $c_{a,i}$ units of a are required. Hence, $k_{a,j}c_{a,i}$ units of j are used-up to produce input a to activity i. The total units of j used-up to produce 1 unit of i is then the sum of the unit j-costs of all the input commodities weighted by the associated input coefficients. Figure 7 depicts the local conservation of real-costs for this example.

In general, the amount of j used-up during the production of 1 unit of i is

$$\sum_{m=1}^{n+2} k_{m,j} c_{m,i} = \mathbf{K}^{(j)} [\mathbf{C}^{(i)}]^{\mathrm{T}},$$

where $\mathbf{C}^{(i)}$ is the ith column of the circular flow.

The total j-cost of input commodities to activity i is simultaneously consumed in the economy as a whole in order to produce unit output of commodity-type i. Hence, the j-cost of commodity-type i is equal to the total input j-cost,

$$k_{i,j} = \mathbf{K}^{(j)} [\mathbf{C}^{(i)}]^{\mathrm{T}}. \tag{15}$$

Let the $1 \times n+2$ row vector $\mathbf{k}_j = [\mathbf{K}^{(j)}]^{\mathrm{T}}$ represent the j-cost of each commodity-type $1 \ldots n+2$. Collecting equations (15) into vector form we can state the complete set of circular flow real-cost equations.

Definition 5.7. The *circular flow real-cost equations* are

$$\mathbf{k}_j = \mathbf{k}_j \mathbf{C}. \tag{16}$$

for $1 \leq j \leq n+2$.

Equation (16) determines the real-cost of all commodity-types in terms of commodity-type j. As each $\mathbf{k}_j$ is determined by the same equation it is clear that $\mathbf{k}_i \propto \mathbf{k}_j$ for all i and j; in other words, the real cost of commodity-types measured in units of i is proportional to the real-cost of commodity-types measured in units of j.

The circular flow is irreducible so a given commodity-type has $n+2$ non-zero real-costs associated with the $n+2$ commodities that enter into its production; that is $\mathbf{K} > \mathbf{0}$.

Lemma 5.2. By equation (16) real-costs are positive and unique for admissible circular flows up to multiplication by a positive number.

Proof. A standard result (for examples in another context, see pages 81 and 310 of [39]). Since 1 is an eigenvalue of $\mathbf{C}$ there is a solution to $\mathbf{k}_j(\mathbf{C} - \mathbf{I}) = \mathbf{0}$. $\mathbf{C} \geq \mathbf{0}$ and irreducible so the associated eigenvector is strictly positive and the only non-negative vector that satisfies the equation (Perron-Frobenius theorems). □

Lemma 5.3. The $n+2$ real-cost solutions are mutually consistent, that is

$$\mathbf{k}_i = k_{j,i}\mathbf{k}_j$$

for all i and j.

Proof. Each $\mathbf{k}_j$, $j = 1, \ldots, n+2$, is determined by the identical equation (16); therefore $\mathbf{k}_i = \alpha_{i,j}\mathbf{k}_j$ for all i and j, where $\alpha_{i,j}$ is a constant of proportionality. As $\mathbf{k}_i = \alpha_{i,j}\mathbf{k}_j$ and $\mathbf{k}_j = \alpha_{j,i}\mathbf{k}_i$ then $\alpha_{i,j} = 1/\alpha_{j,i}$. Since $k_{j,i} = \alpha_{i,j}k_{j,j}$ then $\alpha_{i,j} = k_{j,i}/k_{j,j}$; so $\mathbf{k}_i = (k_{j,i}/k_{j,j})\mathbf{k}_j$. By definition 5.6, $k_{j,j} = 1$ and the conclusion follows. $\square$

Consider production from the perspective of a particular activity $j \in [1, n+2]$, which we shall call the *base activity.* The $n+1$ other activities produce commodities that are directly or indirectly required as inputs to activity j. Hence, from the perspective of the base activity the other commodities are factors of production. For example, in a 4-commodity circular flow, from the perspective of the corn sector, iron, labour and money-capital are factors of production; but from the perspective of the iron sector, corn, labour and money-capital are factors of production, and so forth. In a circular flow all activities are connected and all acts of production are simultaneously acts of consumption. Although the consumption of goods by workers and capitalists is the ultimate goal of economic activity that consumption is a precondition for the reproduction of labour and the advancement of money-capital. Factors of production and the final results of production are relative to the choice of base activity.

Define the index mapping function

$$g_i(x) = \begin{cases} x & \text{when } x < i \\ x+1 & \text{when } x \geq i. \end{cases}$$

Definition 5.8. The j*-inverse* is

$$\mathbf{L}_j = \sum_{i=0}^{\infty} \mathbf{C}[j|j]^i = (\mathbf{I} - \mathbf{C}[j|j])^{-1} = [\lambda_{m,n}],$$

where $\lambda_{m,n}$ represents the physical quantity of the $g_j(m)$th commodity-type needed in the economic system as a whole in order to eventually obtain the availability of 1 unit of the $g_j(n)$th commodity-type as an input to the jth activity. $\mathbf{L}_j$ is a matrix of vertically integrated factor coefficients from the perspective of activity j.

Just as for the Leontief inverse the j-inverse can be better understood by analysing its matrix power-series representation, $\mathbf{L}_j = \sum_{i=0}^{\infty} \mathbf{C}[j|j]^i$. Assume a 'final demand' of unit factors to the base activity, where the factors are all the commodity-types excluding j. $m \in [1, n+1]$ and $n \in [1, n+1]$ index respectively the rows and columns of $\mathbf{L}_j$. The amount of commodity $g_j(m) = m'$ directly required to generate 1 unit of $g_j(n) = n'$ as input to sector j is trivially $\delta_{m',n'}$, where $\delta_{a,b} = 1$ if $a = b$ and $\delta_{a,b} = 0$ otherwise. The unit 'final demands' can therefore be represented by the $n+1 \times n+1$ identity matrix $\mathbf{I}$. To produce 'final demand' $\mathbf{I}$ requires inputs of $\mathbf{IC}[j|j] = \mathbf{C}[j|j]$ (recall that the ith column of $\mathbf{C}[j|j]$ represents the commodity-inputs directly required to produce 1 unit of i, excluding the contribution of activity j). These inputs, in turn, require direct inputs of $\mathbf{C}[j|j]^2$, which, in turn, require direct inputs of $\mathbf{C}[j|j]^3$, and so forth, *ad infinitum.* The vertically integrated factor coefficients are then the sum of the power series $\mathbf{I}+\mathbf{C}[j][j]+\mathbf{C}[j][j]^2+\cdots+\mathbf{C}[j][j]^n$ as $n \to \infty$. The concept of vertical integration, that is 'adding up' real-costs through successive conceptual 'rounds' of production that correspond to the terms of the power-series representation, reveals the interdependencies within the economy, but it does not imply that production within the period is in fact composed of such 'rounds'.

There are $n+2$ j-inverses corresponding to the $n+2$ choices of base activity. The Leontief inverse, defined earlier, represents vertically integrated factor coefficients from the perspective of the worker and capitalist sectors that consume the net product. In contrast, the j-inverse represents vertically integrated factor coefficients from the perspective of a single sector in the circular flow. For example, the $n+1$-inverse represents vertical integration from the perspective of the worker household sector that consumes a part of the net product; similarly, the $n+2$-inverse represents vertical integration from the perspective of the capitalist household sector that consumes the remainder of the net product. Inverses $j = 1, \ldots, n$ represent vertical integration from the perspective of production activities that consume inputs. In the circular flow the distinction between an undistributed net product and other inputs and outputs is absent.

Lemma 5.4. Absolute j-costs, in terms of the j-inverse, are given by

$$\mathbf{k}_j[j] = \mathbf{C}_{(j)}[j]\mathbf{L}_j. \tag{17}$$

Proof. Real-cost equation (16) implies $\mathbf{k}_j(\mathbf{I}-\mathbf{C}) = \mathbf{0}$, which is the homogeneous equation, $\mathbf{k}_j\mathbf{H} = \mathbf{0}$, where $\mathbf{H} = \mathbf{I}-\mathbf{C}$.

$\mathbf{C}$ is admissible and therefore by lemma 5.2 the real-cost solution is undetermined up to a choice of an arbitrary positive number. Consider that the output of activity j is chosen to be the basis commodity. As $\mathbf{H}$ is of rank $n+1$ the system of equations $\mathbf{k}_j\mathbf{H} = \mathbf{0}$ has one equation linearly dependent on the others, which can be removed without consequence. By removing the jth column of $\mathbf{H}$ the homogeneous equation can then be further decomposed into the equivalent system

$$\mathbf{k}_j[j]\mathbf{H}[j|j] + k_{j,j}\mathbf{H}_{(j)}[j] = \mathbf{0},$$

where $\mathbf{k}_j[j]$ is the $1 \times n+1$ row vector of all real costs excluding the basis formed by removing the jth entry from the complete real-cost vector $\mathbf{k}_j$; $\mathbf{H}[j|j]$ is the $n+1 \times n+1$ sub-matrix formed by removing the jth row and jth column from $\mathbf{H}$; $k_{j,j}$ is the real-cost of the basis in terms of units of the basis; and $\mathbf{H}_{(j)}[j]$ is a $1 \times n+1$ row vector equivalent to the jth row of $\mathbf{H}$ excluding its jth element. Relative real-costs can therefore be written in terms of the basis as follows:

$$\mathbf{k}_j[j] = -\mathbf{H}_{(j)}[j]\mathbf{H}[j|j]^{-1}k_{j,j}.$$

As $\mathbf{H} = \mathbf{I} - \mathbf{C}$ then $\mathbf{H}[j|j] = [\mathbf{I} - \mathbf{C}][j|j] = \mathbf{I}[j|j] - \mathbf{C}[j|j] = \mathbf{I} - \mathbf{C}[j|j]$. Hence,

$$\mathbf{k}_j[j] = -\mathbf{H}[j|j](\mathbf{I} - \mathbf{C}[j|j])^{-1} k_{j,j}.$$

Also, $\mathbf{H}_{(j)}[j] = [\mathbf{I} - \mathbf{C}]_{(j)}[j] = \mathbf{I}_{(j)}[j] - \mathbf{C}_{(j)}[j] = \mathbf{0} - \mathbf{C}_{(j)}[j] = -\mathbf{C}_{(j)}[j]$. Hence,

$$\mathbf{k}_j[j] = \mathbf{C}_{(j)}[j](\mathbf{I} - \mathbf{C}[j|j])^{-1} k_{j,j}, \tag{18}$$

expressed in terms of the variable $k_{j,j}$, which by definition 5.6 is 1. □

Opocher [55], following a suggestion by Harrod, examines real-costs (via a different route to the one adopted here) in Sraffa's closed equations. He makes the important observation that real-costs have a dual interpretation. As $k_{i,j}$ units of j are used-up during the production of unit i, and all amounts productively consumed are received via market exchanges, then real-costs are also market rates of exchange. So the real-cost of commodity i in terms of commodity j is the amount of j per unit of i that enters the market. The following lemma expresses the fact that this relationship is reciprocal.

Lemma 5.5. The j-cost of i is the reciprocal of the i-cost of j; that is

$$k_{i,j} = \frac{1}{k_{j,i}}, \tag{19}$$

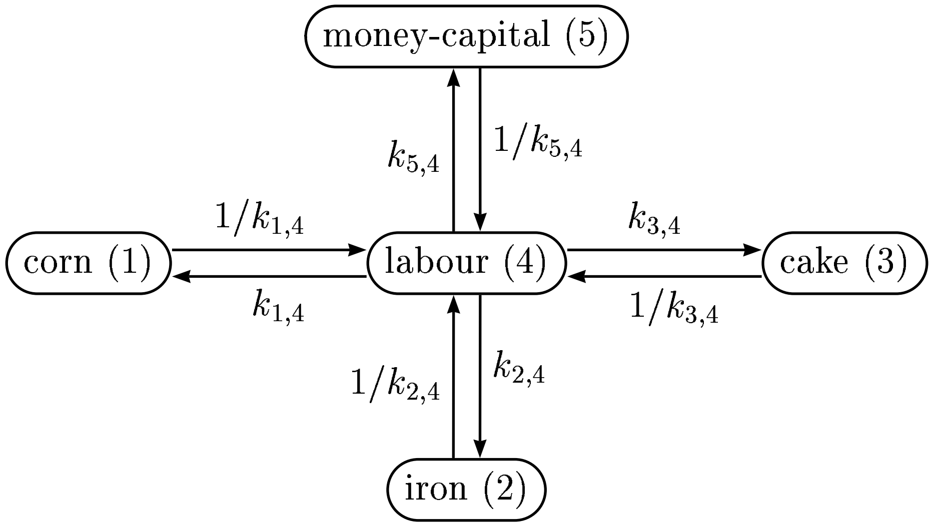
for all i and j.

Proof. By lemma 5.3, $k_{k,i} = k_{j,i} k_{k,j}$ and $k_{k,j} = k_{i,j} k_{k,i}$; combining $k_{i,j} = 1/k_{j,i}$. □

For example, if $k_{i,n+1}$ units of i enter the market per unit of labour then $1/k_{i,n+1} = k_{n+1,i}$ units of labour enter the market per unit of i. Assume the 'own use' of commodity-type i by activity i is mediated by a market exchange. Then $k_{i,i} = 1$ expresses the fact that every unit of i produced is exchanged in the market.

Real-costs in the circular flow therefore support (at least) two interpretations: the *vertical integration interpretation*, in which the real-cost of i in terms of j represents the total amount of j used-up in the economy as a whole during the production of unit i; and the *market exchange interpretation*, in which real-cost is the amount of j per unit i that enters the market. Let's analyse the meaning of these interpretations in a little more depth.

Ellerman [14] analyses equilibrium prices in closed 'market graphs' and notes that 'there exists a system of prices for the commodities such that the given exchange rates are the price ratios if and only if the exchange rates are arbitrage-free (in the sense that they multiply to one around any circle)' and relates this principle to many kinds of conservative systems, including Kirchoff's Voltage Law in electrical circuits. The principle was first formulated by Cournot (1838) for rates of currency exchange [11], and was taken up by Walras in his analysis of equilibrium ([35], ch.2). Krause's analysis of equal exchange in an 'exchange-structure' ([35], ch.2) employs the same non-arbitrage principle. Cockshott and Cottrell [8] make an equivalent observation and point out that the metric of commodity-exchange implies there is an underlying linear conservation law.



$$\mathbf{K} = \begin{bmatrix} 1 & k_{1,2} & k_{1,3} & k_{1,4} & k_{1,5} \\ \frac{1}{k_{1,2}} & 1 & k_{2,3} & k_{2,4} & k_{2,5} \\ \frac{1}{k_{1,3}} & \frac{1}{k_{2,3}} & 1 & k_{3,4} & k_{3,5} \\ \frac{1}{k_{1,4}} & \frac{1}{k_{2,4}} & \frac{1}{k_{3,4}} & 1 & \frac{1}{k_{5,4}} \\ \frac{1}{k_{1,5}} & \frac{1}{k_{2,5}} & \frac{1}{k_{3,5}} & k_{5,4} & 1 \end{bmatrix}.$$

FIGURE 8. **Real-cost graph for a 3-sector economy with labour as the basis**. The complete real cost graph is fully connected. Here were depict only the edges to and from the labour sector, where $\mathbf{k}_4 = [\mathbf{K}^{(4)}]^{\mathrm{T}}$. The labour-cost of i, $k_{i,4}$, is the reciprocal of the i-cost of labour, $k_{4,i} = 1/k_{i,4}$. The real-costs around any closed loop multiply to 1.

In our context real-costs also satisfy this conservation property. A real-cost graph is associated with every circular flow. Under the market exchange interpretation the edges of the real-cost graph are market rates of exchange. Figure 8 depicts an example real-cost graph for a 3-sector economy (plus 2 household sectors).

Definition 5.9. A *closed loop* in the real-cost graph is a sequence of sector indices, $S = \{b, \ldots, b\}$, where index b only appears once at the beginning of the sequence and once at the end of the sequence.

If the product of the market exchange rates around any closed loop in the real-cost graph is greater than 1 then real-cost profits are possible. For example, assume the labour-cost of iron is $k_{2,4} = 2$ hours per tonne and the iron-cost of labour is $k_{4,2} = 1.5$ tonnes per hour; then $k_{2,4}k_{4,2} = 3 > 1$. A market agent that has 1 tonne of iron can exchange it for 2 hours of labour, and then exchange the 2 hours of labour for 3 tonnes of iron, ending up with more iron than at the start [8].

Under the vertical integration interpretation the edges of the real-cost graph are commodity transformation rates. In this case, if the product of the transformation rates around any closed loop is greater than 1 then stock accumulation is possible.

That real-costs around a closed loop multiply to 1 is therefore an equilibrium condition.

Proposition 5.1. Real-costs around a closed loop multiply to 1.

Proof. See appendix B. □

Sraffa's analysis of real-cost properties in his closed model of Chapter 1 is inadequate, being restricted to, 'while in the two-industry system the amount of iron used in wheat-growing was necessarily of the same value as the amount of wheat used in iron-making, this, when there are three or more products, is no longer necessarily true of any pair of them. Thus in the last example there is no such equality and replacement can only be effected through triangular trade' [75]. But as we have shown the reciprocal relationship between real-costs obtains in higher-dimensional economies. The concept of real-cost does not feature in PCMC after this point. Marx's examination of the expanded and general form of value in the early chapters of Volume I of *Capital* [44] is an example of real-cost analysis.

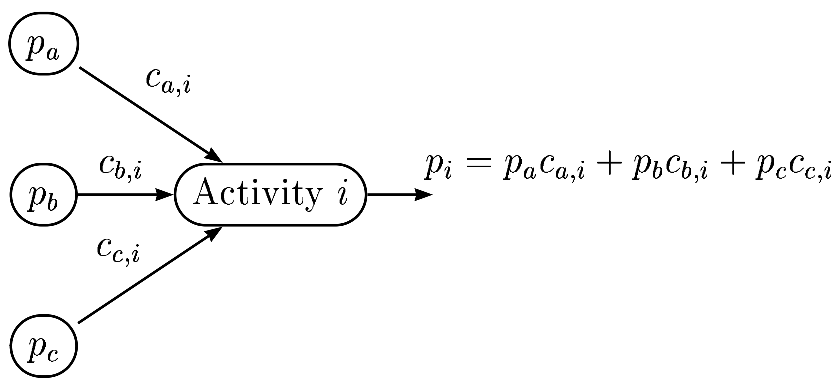


FIGURE 9. **Local price conservation**. The price of commodity i is the weighted sum of the unit prices of all its input commodities, in this case a, b and c.

5.2. **Price accounting in the circular flow.**

Definition 5.10. The *circular flow price equation* is

$$\mathbf{p}^\circ = \mathbf{p}^\circ \mathbf{C}, \tag{20}$$

where $\mathbf{p}^\circ = [p_i]$ is a $1 \times n + 2$ row vector of prices.

Price equation (20) defines stationary, equilibrium prices. It expresses the condition that for each activity the total price of commodity inputs equals the price of the commodity output. The costs and revenues for all activities are in perfect balance and monetary value is conserved in production. Figure 9 depicts the local conservation of monetary value.

Note that the circular flow price equation is a closed input-output model, identical in form but different in structure to Leontief's closed model and Sraffa's subsistence, or 'no surplus', equations. Note also that the form of the price equation, $\mathbf{p}^\circ = \mathbf{p}^\circ\mathbf{C}$, is identical to the form of the real-cost equations $\mathbf{k}_j = \mathbf{k}_j\mathbf{C}$ $(j = 1, \ldots, n+2)$.

Lemma 5.6. Equilibrium prices are positive and unique for admissible circular flows up to multiplication by a positive number.

Proof. Prices are the eigenvector associated with the unit eigenvalue of $\mathbf{C}$. The proof is identical to lemma 5.2. □

Lemma 5.7. Relative prices solutions in the circular flow are

$$\mathbf{p}^\circ[j] = \mathbf{C}_{(j)}[j]\mathbf{L}_j p_j^\circ \tag{21}$$

for $j = 1, \ldots, n+2$.

Proof. The price equation is identical in form to any of the $n+2$ real-cost equations; hence, the conclusion follows via a similar argument to lemma 5.4. □

Leontief notes that prices in a closed model equal total unit costs, 'but this is the "law of value" of the so-called objective value theory' [41]. Sraffa remarks that prices in his closed equations 'spring directly from the methods of production' [75], but does not elucidate further. Opocher [55] has recently shown that relative prices in Sraffa' subsistence equations reflect 'physical real-costs', a concept that 'Sraffa was particularly interested in the early times in which the [closed] equations were first formulated'. Circular flow prices also have this property.

Theorem 5.8 (Circular flow prices are proportional to real-costs)**.** The price of commodity i equals the j-cost of i multiplied by the price of commodity j:

$$p_i = k_{i,j}p_j,$$

for all i and j. In vector notation, $\mathbf{p}^\circ[j] = \mathbf{k}_j[j]p_j^\circ$.

Proof. Substitute equation (17) (lemma 5.4) into equation (21) (lemma 5.7) to get $\mathbf{p}^\circ[j] = \mathbf{k}_j[j]p_j^\circ$. □

For example, the price of commodity-type i is equal to the price of any commodity-type j multiplied by the amount of j that is used-up in the economy as a whole in order to produce unit i.

The simple, value-theoretic *principle of real-cost* governs the circular flow in a state of self-replacing equilibrium.

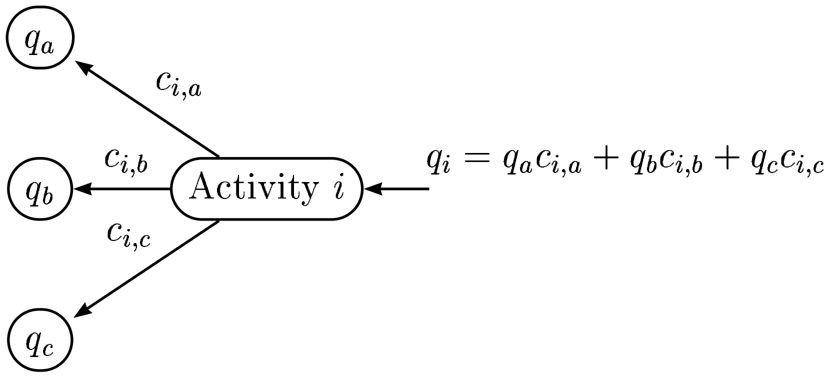


FIGURE 10. **Local quantity conservation**. The production of q_i units of commodity i is distributed to other activities, in this case a, b and c, in definite proportions.

5.3. **Quantity accounting in the circular flow.** The analysis of quantity solutions is included for completeness.

Definition 5.11. The *circular flow quantity equation* is

$$\mathbf{q}^\circ = \mathbf{q}^\circ \mathbf{C}^{\mathrm{T}}, \tag{22}$$

where $\mathbf{q}^\circ = [q_i]$ is a $1 \times n+2$ row vector of quantities.

Quantity equation (22) defines stationary, equilibrium quantities. It expresses the condition that for each activity the total quantity of outputs equals the total quantity of inputs to other activities. Figure 10 depicts the local conservation of quantities in exchange.

Lemma 5.9. Equilibrium quantities are positive and unique for admissible circular flows up to multiplication by a positive number.

Proof. The proof is similar to lemma 5.2, except we use the transpose of $\mathbf{C}$. Quantities are the eigenvector associated with the unit eigenvalue of $\mathbf{C}^{\mathrm{T}}$. □

Lemma 5.10. Relative quantity solutions in the circular flow are

$$\mathbf{q}^\circ[j] = [\mathbf{C}^{(j)}]^{\mathrm{T}}[j]\mathbf{L}_j^{\mathrm{T}} q_j^\circ \tag{23}$$

for $j = 1, \ldots, n+2$.

Proof. The quantity equation is identical in form to any of the $n+2$ real cost equations, except $\mathbf{C}$ is replaced by its transpose; hence, the conclusion follows by a similar argument to lemma 5.4 with the additional observations that the j-inverse of $\mathbf{C}^{\mathrm{T}}$ is the transpose of the j-inverse of $\mathbf{C}$ and $[\mathbf{C}^{\mathrm{T}}]_{(j)} = [\mathbf{C}^{(j)}]^{\mathrm{T}}$. □

6. Prices of production are proportional to real-costs

We now get to the crux of the argument: *Sraffian prices are identical to circular flow prices* and therefore are proportional to real-costs. This fact is hidden in Sraffa's surplus representation of simple reproduction.

Recall that $\mathbf{p}$ denotes Sraffian prices and $\mathbf{p}^\circ$ denotes circular flow prices.

Theorem 6.1 (Sraffian prices are circular flow prices)**.**

$$[\mathbf{p}, w, r] = \mathbf{p}^\circ.$$

Proof. By equation (21), circular flow prices, when expressed in terms of the rate of profit, are

$$\mathbf{p}^\circ[n+2] = \mathbf{p}^\circ[n+2]\mathbf{C}[n+2|n+2] + \mathbf{C}_{(n+2)}[n+2]p^\circ_{n+2} \tag{24}$$

$$= \mathbf{p}^\circ[n+2]\begin{bmatrix} \mathbf{A}^\circ & [\bar{\mathbf{w}}^\circ]^{\mathrm{T}} \\ \mathbf{l}^\circ & 0 \end{bmatrix} + \begin{bmatrix} \bar{\mathbf{m}}^\circ & 0 \end{bmatrix} r^\circ, \tag{25}$$

where $r^\circ = p^\circ_{n+2}$ by definition. Let $\mathbf{p}^{\mathrm{c}} = \mathbf{p}^\circ[n+1, n+2]$ be the prices of all commodities, excluding the wage and profit rate. Then (25) can be written as

$$\begin{bmatrix} \mathbf{p}^{\mathrm{c}} & w^\circ \end{bmatrix} = \begin{bmatrix} \mathbf{p}^{\mathrm{c}}\mathbf{A}^\circ + \mathbf{l}^\circ w^\circ & \mathbf{p}^{\mathrm{c}}[\bar{\mathbf{w}}^\circ]^{\mathrm{T}} \end{bmatrix} + \begin{bmatrix} \bar{\mathbf{m}}^\circ r^\circ & 0 \end{bmatrix},$$

where $w^\circ = p^\circ_{n+1}$ by definition. Hence, we get two equations

$$\mathbf{p}^{\mathrm{c}} = \mathbf{p}^{\mathrm{c}}\mathbf{A}^\circ + \bar{\mathbf{m}}^\circ r^\circ + \mathbf{l}^\circ w^\circ \tag{26}$$

$$w^\circ = \mathbf{p}^{\mathrm{c}}[\bar{\mathbf{w}}^\circ]^{\mathrm{T}}. \tag{27}$$

By definition of the money-capital constraint, $\bar{\mathbf{m}}^\circ = \mathbf{p}^{\mathrm{c}}\mathbf{A}^+$, where $\mathbf{A}^+ = \mathbf{A}^\circ + [\mathbf{w}^\circ]^{\mathrm{T}}\mathbf{l}^\circ$. Substitute for $\bar{\mathbf{m}}^\circ$ into equation (26) to get

$$\mathbf{p}^{\mathrm{c}} = \mathbf{p}^{\mathrm{c}}\mathbf{A}^\circ + \mathbf{l}^\circ w^\circ + (\mathbf{p}^{\mathrm{c}}\mathbf{A}^\circ + \mathbf{p}^{\mathrm{c}}[\bar{\mathbf{w}}^\circ]^{\mathrm{T}}\mathbf{l}^\circ)r^\circ. \tag{28}$$

Substitute equation (27) into (28) to get

$$\mathbf{p}^{\mathrm{c}} = (\mathbf{p}^{\mathrm{c}}\mathbf{A}^\circ + \mathbf{l}^\circ w^\circ)(1 + r^\circ), \tag{29}$$

which is a Sraffian price equation (2).

Consider the Sraffian system Ψ. By theorem 5.1 Ψ determines a circular flow $\mathbf{C}_\Psi$. Hence, equation (29) corresponding to $\mathbf{C}_\Psi$ is identically the price equation of the Sraffian system Ψ and $\mathbf{p} = \mathbf{p}^{\mathrm{c}}$.

Also, $w = \mathbf{p}\bar{\mathbf{w}}^{\mathrm{T}} = \mathbf{p}^{\mathrm{c}}[\mathbf{w}^\circ]^{\mathrm{T}} = w^\circ$.

Similarly, by equation (21), circular flow prices, when expressed in terms of the wage, are

$$\begin{aligned}
\mathbf{p}^\circ[n+1] &= \mathbf{p}^\circ[n+1]\mathbf{C}[n+1|n+1] + \mathbf{C}_{(n+1)}[n+1]p^\circ_{n+1} \\
&= \mathbf{p}^\circ[n+1]\begin{bmatrix} \mathbf{A}^\circ & [\bar{\mathbf{c}}^\circ]^{\mathrm{T}} \\ \bar{\mathbf{m}}^\circ & 0 \end{bmatrix} + \begin{bmatrix} \mathbf{l}^\circ & 0 \end{bmatrix} w^\circ \\
\begin{bmatrix} \mathbf{p}^{\mathrm{c}} & r^\circ \end{bmatrix} &= \begin{bmatrix} \mathbf{p}^{\mathrm{c}}\mathbf{A}^\circ + \bar{\mathbf{m}}^\circ r^\circ & \mathbf{p}^{\mathrm{c}}[\bar{\mathbf{c}}^\circ]^{\mathrm{T}} \end{bmatrix} + \begin{bmatrix} \mathbf{l}^\circ w^\circ & 0 \end{bmatrix}.
\end{aligned}$$

Hence, $r^\circ = \mathbf{p}^{\mathrm{c}}[\bar{\mathbf{c}}^\circ]^{\mathrm{T}}$. By lemma 4.8, $r = r^\circ$.

Combining, $[\mathbf{p}, w, r] = \mathbf{p}^\circ$, as required. □

For completeness note that Sraffian quantities are circular flow quantities.

Theorem 6.2 (Sraffian quantities are circular flow quantities)**.**

$$[\mathbf{q}, L, M] = \mathbf{q}^{\circ}.$$

Proof. See appendix C (the proof is similar to the proof of theorem 6.1). □

It immediately follows from theorem 6.1 that Sraffian prices of production are proportional to real-costs.

Corollary 6.3 (Prices of production are proportional to real-costs).

$$[\mathbf{p}, w, r][j] = \mathbf{k}_j[j]p_j, \tag{30}$$

for all $j \in [1, n+2]$.

Proof. By theorem 6.1, Sraffian prices of production are circular flow prices, $[\mathbf{p}, w, r] = \mathbf{p}^{\circ}$. By theorem 5.8 circular flow prices are proportional to real-costs, $\mathbf{p}^{\circ}[j] = \mathbf{k}_j[j]p^{\circ}_j$. Hence, prices of production are proportional to real-costs, $[\mathbf{p}, w, r][j] = \mathbf{k}_j[j]p_j$. □

Once the real consumption of workers and capitalists is specified then Sraffa's surplus equations implicitly define a circular flow (theorem 5.1). Both representations describe the same economic state-of-affairs and the price and quantity solutions are identical (theorems 6.1 and 6.2). The circular flow $\mathbf{C}_{\Psi}$ and the Sraffian system Ψ are different representations of the same theory of prices of production. Henceforth, we will freely swap between the Sraffian point-of-view, Ψ, and the monetary-production, circular flow point-of-view, $\mathbf{C}_{\Psi}$.

A circular flow $\mathbf{C}_{\Psi}$ corresponding to the Sraffian system Ψ can only be constructed if the real distribution of the net product is specified. It is natural to specify this information when Sraffa's surplus equations are interpreted to represent simple reproduction in a state of self-replacing, 'long-period' equilibrium (and natural generalisations). 'Non-equilibrium' interpretations of Sraffa's system, briefly mentioned in section 3.4, in which the real distribution of the net product is not specified, are therefore not ruled out by this result. The principle of (realised) real-cost may be absent in such cases. Yet whether an equilibrium or non-equilibrium interpretation of Sraffa's equations is more faithful to his intended theoretical project is not relevant from the point of view of understanding simple reproduction in a state of self-replacing equilibrium. The point is that, if simple reproduction (and natural generalisations) are the object of study then the circular flow and Sraffa's system are equivalent.

The circular flow avoids the theoretical problems of Sraffa's open system. Prices are always determinate in the circular flow because it is a fully connected system that does not break apart into basic and non-basic sub-techniques. The rate of profit of the system is necessarily a global property of the economy, rather than a local property of the basic sub-technique, in virtue of the causal connections between all activities. There is no problem of self-reproducing non-basic sub-techniques; hence additional assumptions do not need to be introduced to fix Sraffa's price theory. Also, the objective approach of counting inputs and output applies equally to the real wage in the circular flow. The subjective element that enters the analysis of the surplus equations, due to the need to classify wage goods as necessary or contingent for reproduction, is avoided. The production of a surplus is not a novel distributional event that is unconnected to the next round of production – all the surplus is consumed, and all of it is 'necessary' in the circular flow. In contrast, if the real distribution of the net product is not specified then the problems of indeterminate prices and subjectivism remain unresolved in Sraffa's system.

The circular flow provides a new perspective on some standard Sraffian themes. The construction of a circular flow from Sraffa's system can be reversed by 'opening' the circular flow. A total of $\frac{1}{2}(n+2)(n+1)$ 'surplus' equation systems, consisting of a price and quantity equation, can be constructed, one for each pair of activities whose joint inputs are to be considered undistributed. The standard Sraffian system corresponds to the special case when activities $n+1$ and $n+2$ are chosen for 'opening'. The other systems are *non-standard Sraffian systems*, in which a non-standard 'surplus' is consumed by the selected pair of activities. The standard and non-standard systems have mutually consistent price and quantity solutions and are thus different representations of the same economic state-of-affairs. Sraffa's distinction between basic and non-basic commodities is relative to the choice of 'opened' activities. The set of basics in a non-standard system in general differs from the set of basics of the standard system. 'Non-standard' standard commodities can be constructed that have the same properties that hold for the standard commodity but with respect to bi-lateral price trade-offs rather than changes in the distribution of income. And so on.

The circular flow is a monetary-production economy and, in contrast to Sraffa's surplus equations, money-capital is as a factor of production with an associated cost. The rate of profit is the price of nominal capital. That money-capital has a real-cost does not imply that capitalists either justly or unjustly receive profit income. Input-output relations only describe what pertains. And under capitalist production, or at least the theoretical idealisation of it considered here, capitalists operate a monopoly on capital and receive a return for the act of supplying funds to production. Money-capital is a factor of production because it is a socially necessary input to production. And it has a real-cost because a condition of its advancement is the consumption of a commodity bundle.

The circular flow representation discloses the value-theoretic principle of real-cost. In contrast, Sraffa's surplus representation obscures it because the circular flow is interrupted. Opocher, for example, shows that the concept of real-cost that applies to Sraffa's closed equations does not transfer to Sraffa's open equations. The real-cost of a basic in terms of a non-basic is null because 'there is no *exchange* between possessors of basics and non-basics' (and hence no market exchange rates). Opocher concludes that Sraffa 'did well to ignore it [the principle of real-cost] in his analysis of the self-replacing state' [55]. I conclude the opposite – that Sraffa's surplus representation is incomplete precisely because it fails to disclose this principle. Once the surplus is fully distributed real-cost is fully defined. The surplus is still there in the circular flow but it no longer has an exceptional status as an undistributed output. Hence, *contra* Sraffa, the transition from subsistence (chapter 1 of PCMC) to surplus equations (chapter 2) need not imply that the economic object under study changes such that 'the system becomes self-contradictory' ([75], p. 6) and, for example, a new class of non-basic good appears. I think a different interpretation of this transition is possible. Kurz and Salvadori [38] have partially documented how Sraffa's concept of objectivism subtly shifted as he began to study an undistributed surplus. The transition from subsistence to surplus is an implicit switch in Sraffa's problematic away from an analysis of the *necessary* relations that obtain in a state of self-replacing equilibrium toward an analysis of the *contingent* consequences of changes in the distribution of income, from an early objectivist problematic perhaps best represented by Petty's methodological injunction to construct theories 'in Terms of Number, Weight, or Measure; and to use only Arguments of Sense, and to consider only such Causes, as have visible Foundations in Nature' (*Political Arithmetick*, 1690) toward a Ricardian problematic that views political economy as essentially 'an enquiry into the laws which determine the division of produce of industry amongst the classes that concur in its formation' (*Letter to Malthus*, 1820 [61]). The interruption of the circular flow in order to theorise income distribution, absent a proper dynamic theory, is perhaps the theoretical reason why Sraffa was unable to sustain the strict objectivism of Petty and the value-theoretic principle of real-cost in his surplus equations.

The 'self-contradiction' that Sraffa alludes to in the transition from closed to open model is the breaking of the exchange symmetries that obtain between real-costs in closed models (see in particular lemma 5.5). The symmetry is broken due to the production of an *undistributed* surplus. The open representation allows nominal changes in income distribution to be studied. However, if the object of study is an economy with a *distributed* surplus, such as a state of self-replacing equilibrium, then the Sraffian open model is equivalent to a closed model, in which money-capital is a factor of production. In such a case, an open model representation of self-replacing equilibrium is misleading because it suggests that the production of a surplus continues to break real-cost exchange symmetries. But in this context there are no changes in income distribution, because once the composition of the net product is specified and distributed to workers and capitalists, there are no contingent consequences of varying the income distribution to examine. Once the surplus is fully distributed and a full equilibrium state is obtained then the real-cost exchange symmetries return.

To summarise, in an open model prices are determined once the nominal distribution of income is specified. Similarly, real-costs are determined once the real distribution of income is specified. Both closures define a circular flow representation of an economy in which the value-theoretic principle of real-cost obtains. Specifically, profit-equalising prices of production are proportional to real-costs.

The classical concept of labour-value is a real-cost with labour as the basis. We'd therefore expect the principle of real-cost to have implications for the classical labour theory of value. We now turn to applying the principle to an old and intractable problem.

Part 2. Marx's conservation law

7. The transformation problem

7.1. **The classical context of the transformation problem.** The classical economists, Adam Smith and David Ricardo, observed that human labour in general is the animate and intentional cause of economic activity. In consequence they adopted, more or less consistently, a labour theory of value in which the labour-cost or *labour value* of reproducible goods determines their price, absent supply and demand fluctuations. Ricardo consistently tried to apply the principle of the law of labour value in his *Principles* (1817) [60]. He considered the development of the productivity of labour as the ultimate cause of price changes. The extensive application of machinery in 19th century England compelled Ricardo to consider how fixed capital, that is money-capital tied-up in the form of durable and relatively long lasting machinery, tools, equipment, buildings, and so forth, might modify the law of labour value [66]. Ricardo reasoned, with the aid of numerical sketches, that prices could not be simple quantitative measures of labour-time under conditions of equal returns to capital invested. But unlike Smith, who concluded that the determination of value by labour-time was not applicable to 'civilised' times in which fixed capital is employed [74], Ricardo concluded that although the principle must be 'considerably modified' and subject to exceptions [60] it nonetheless remained a valid basis from which to study the capitalist economy. Yet Ricardo acknowledged in his correspondence that the modifications had introduced a contradiction into the theory of value. The contradiction contributed to the dissolution of the Ricardian school of economic thought in the 19th century [66].

Marx inherited this classical problematic within the much broader context of historical materialism. For Marx, 'labour is the substance, and the immanent measure of value, but has itself no value' [44]. According to Marx, classical political economy 'never once asked the question why labour is represented by the value of its product and labour-time by the magnitude of that value' [44]. Marx argues, in the early chapters of Volume I of *Capital* [44], that the causal powers of social labour, at a certain stage of historical development, are of necessity represented by money in a system of generalised commodity exchange. His theory of the law of value, an extension and modification of Smith's 'invisible hand' [74], explains how the total labour of a society is divided and allocated to different branches of production via the market, 'behind the backs' of self-interested producers. The exchange of commodities at prices that deviate from values is the mechanism by which social labour-time is transferred from one sector of production to another. When prices equal values the division of labour has reached an equilibrium that satisfies social demand. Rubin summarises that 'the law of value is the law of equilibrium of the commodity economy' ([65], p. 67; and see p. 336 of [44], p. 178 of [45] and [82]). In Marx's version of the labour theory of value, labour-time is measured and directed by the operation of the law of value, an objective social mechanism brought into being as an unintended consequence of production and market exchange. Hence, unlike the classical authors, who adopted labour cost as a natural unit of measurement or conventional yardstick, Marx's 'critique of political economy was *not* one which involved him finding a "constant" in terms of which everything could be quantified but of establishing the laws of mediation through which the "essence" of phenomena manifested itself as "appearance"' [58]. Marx, particularly in the early and difficult parts of Volume I, constructed a theory to explain how the causal power of social labour in fact manifests as 'dazzling money-forms' [44], in which 'relations among *people* acquire the form of equalisation among *things*' ([65], p.16), a kind of anthropological approach that identifies the characteristic social fetishes, or ideological illusions, of commodity exchange.

In Volume I of *Capital* Marx adopts the theoretical simplification that commodities exchange at prices proportional to labour values. In consequence, there is a constant of proportionality, more recently named the monetary expression of labour-time (MELT) [18, 19, 20, 81], which translates between labour value, measured in units of labour-time, and price, measured in units of currency. Marx defines the value (labour-cost) of labour-power as 'the value of the means of subsistence necessary for the maintenance of the labourer' ([44], p.167). But workers do not receive the whole of the net product. A part is distributed to the capitalist class in the form of profits. Marx demonstrates that total profit in a closed economy is the monetary representation of the total unpaid labour of the working class, or surplus-value. Hence capitalism, although progressive in virtue of its development of the productive powers of social labour, is exploitative. Capitalism shares this trait with previous class-based societies, for example feudal economic arrangements, in which the corvee peasant works a set number of days per year gratis for the local landlord. But under capitalism the exploitative labour-time transfers between capitalists and workers are obscured by the apparent equitable money transfers that occur in the marketplace.

> The wage form thus extinguishes every trace of the division of the working day into necessary labour and surplus-labour, into paid labour and unpaid labour. All labour appears as paid labour. Under the corvee system it is different. There the labor of the serf for himself, and his compulsory labour for the lord of the land, are demarcated very clearly both in space and time. [44].

Marx's theory of value directly challenges the key ideological premise of capitalism that wages are a proper reward of labour.

Marx's demonstration of the identity of profit and unpaid labour relies on the assumption of price-value proportionality. Yet Smith and Ricardo had argued, more or less clearly, that the law of value and the law of uniform profit are incompatible. Marx needed to resolve the contradiction of the classical labour theory of value to maintain the quantitative connection between the dual accounting systems of price and labour-time that formed the material basis of his account and critique of capitalist economic relations.

Marx turned to the problem in his unfinished notes published as Volume III of *Capital*. In chapter 9 Marx presents the classical contradiction as it manifests in his theoretical framework. For Marx, the labour value, θ, of a commodity consists of constant capital, c, which represents means of production used-up, variable capital, v, which represents the workers wages, and surplus-value, s, which represents the unpaid labour-time. Assume there is no fixed capital, so constant capital consists purely of circulating capital, and that, following Lippi [42], the economy consists of just two activities. Then the labour values, θ_1 and θ_2 of each commodity are

$$\theta_1 = c_1 + v_1 + s_1, \tag{31}$$

$$\theta_2 = c_2 + v_2 + s_2. \tag{32}$$

In Marx's theory, only labour creates surplus-value; hence, the amount of surplus-value produced by each activity depends on the variable, not the constant, capital. Marx assumes 'that the capitals in the different spheres of production annually realise the same quantities of surplus-value proportionate to the magnitude of their variable parts' ([45], p.154); in other words, the rates of surplus value are equal,

$$\frac{s_1}{v_1} = \frac{s_2}{v_2}.$$

But capitalists earn profit on the whole of the capital supplied. So if prices are proportional to labour values then the rates of profit in the two sectors are

$$r_1 = \frac{s_1}{c_1 + v_1} = \frac{s_1}{v_1} \frac{1}{\frac{c_1}{v_1} + 1}$$

and

$$r_2 = \frac{s_2}{c_2 + v_2} = \frac{s_2}{v_2} \frac{1}{\frac{c_2}{v_2} + 1}.$$

Since we have assumed equal rates of surplus-value then the rates of profit are equal only if the 'organic composition of capitals', that is the ratios c_1/v_1 and c_2/v_2, are equal. But there is no economic reason to think that the ratio of the value of the means of production to the value of labour in both sectors should be equal. Hence, 'in the different spheres of production with the same degree of exploitation, we find considerably different rates of profit corresponding to the different organic composition of these capitals' ([45], p.155). And 'capitals of equal size produce commodities of unequal *values* and therefore yield *unequal surplus-values or profits*, because value is determined by labour-time, and the amount of labour-time realised by a capital does not depend on its absolute size but on the size of the variable capital, the capital laid out in wages' ([46], p. 190). The law of value seems to contradict the law of uniform profits.

Marx proposed a solution: 'the rates of profit prevailing in the various branches of production are originally very different' ([45], p. 158) but the different rates 'are equalised by competition to a single general rate of profit' ([45], p. 158). At which point, 'although in selling their commodities the capitalists of various spheres of production recover the value of the capital consumed in their production, they do not secure the surplus-value, and consequently the profit, created in their own sphere by the production of these commodities. What they secure is only as much surplus-value, and hence profit, as falls, when uniformly distributed, to the share of every aliquot part of the total social capital from the total social surplus-value, or profit, produced in a given time by the social capital in all spheres of production' ([45], p.158). So although the rate of surplus-value is in fact uniform that rate at which it is appropriated in the form of profit by individual capitals is non-uniform. In our example the general rate of profit is

$$r = \frac{s_1 + s_2}{c_1 + c_2 + v_1 + v_2}, \tag{33}$$

and the 'prices of production' that effect the redistribution of surplus-value due to the formation of a uniform rate of profit are

$$p_1 = c_1 + v_1 + r(c_1 + v_1), \tag{34}$$

and

$$p_2 = c_2 + v_2 + r(c_2 + v_2).$$

'Hence, the price of production of a commodity is equal to its cost-price plus the profit, alloted to it in per cent, in accordance with the general rate of profit, or, in other words, to its cost-price plus the average profit' ([45], p.157).

Marx's 'prices of production' differ from the original labour values. In general $p_1 \neq \theta_1$ and $p_2 \neq \theta_2$. According to Marx, 'one portion of the commodities is sold above its value in the same proportion in which the other is sold below it. And it is only the sale of the commodities at such price that enables the rate of profit for capitals [to be uniform], regardless of their different organic composition' ([45], p.157). The equalisation of the rate of profit requires that prices diverge from values. And it is easy to verify that Marx's 'prices of production' equal labour values only in the special case of equal organic compositions of capital.

According to Marx, however, the labour theory of value continues to hold in the aggregate because prices are merely transformed values and profit is redistributed surplus-value [32]. The amount of surplus-labour produced by workers is not affected by its appropriation in the form of capitalist profits. So Marx postulated three conservation rules or 'laws' concerning the aggregates in his transformation of values to 'prices of production'. Assume, without loss of generality, that the MELT is 1, then Marx contends that 'the sum of the profits in all spheres of production must equal the sum of the surplus-values, and the sum of the prices of production of the total social product equal the sum of its value' ([45], p. 173). That is, (i) the 'sum of the prices of production of all commodities produced in society – the totality of all branches of production – is equal to the sum of the values' ([45], p.160), (ii) total profit is equal to total surplus-value (Ch. 9 in [45] and the extensive quotations in [51]), and (iii) the rate of profit is equal to the ratio of total surplus-value to total capital ([56], p. 129–130), which is the conservation rule in equation (33) that Marx employs to form the general rate of profit. Marx's aggregate equalities do hold after his transformation; that is, conservation of the value of gross output:

$$p_1 + p_2 = (c_1 + v_1 + c_2 + v_2)(1 + r) = \theta_1 + \theta_2,$$

and conservation of surplus-value:

$$r(c_1 + v_1) + r(c_2 + v_2) = s_1 + s_2.$$

Although price and value diverge, due to law of uniform profits, aggregate price and value magnitudes are identical, due to the law of value. The classical contradiction is thereby resolved. Marx's considered that his transformation procedure also explained why the origin of profit in labour is obscured by capitalist competition. 'Since in the rate of profit the surplus-value is calculated in relation to the total capital and the latter is taken as its standard measurement, the surplus-value itself appears to originate from the total capital, uniformly derived from all its parts, so that the organic difference between constant and variable capital is obliterated in the conception of profit. Disguised as profit, surplus-value actually denies its origin, loses its character, and becomes unrecognisable' ([45], p.167). In Marx's theory, capitalist economic relations are a persistent cause of mistaken ideas about how the economy functions.

So far so good. But Marx straightaway observes that, 'we had originally assumed that the cost-price of a commodity equalled the *value* of the commodities consumed in its production. But for the buyer the price of production of a specific commodity is its cost-price, and may thus pass as a cost-price into the prices of other commodities. Since the price of production may differ from the value of a commodity, it follows that the cost-price of a commodity containing the price of production of another commodity may also stand above or below that portion of its total value derived from the value of the means of production consumed by it. It is necessary to remember this modified significance of the cost-price, and to bear in mind that there is always the possibility of an error if the cost-price of a commodity in any particular sphere is identified with the value of the means of production consumed by it. Our present analysis does not necessitate a closer examination of this point' ([45], p.165).

The 'prices of production' are calculated on the basis of untransformed cost-prices. But the untransformed prices are market prices that enter the cost-price of commodities. In other words, Marx transformed the output prices but not the input prices. Marx recognises, therefore, that his transformation procedure is, in some sense, incomplete, although the precise nature of this incompleteness is open to interpretation due to the unfinished nature of chapter 9. Lippi remarks that Marx knows 'the magnitudes on the basis of which surplus-value has been redistributed – that is, capital advanced, measured in value – are not identical to the prices at which elements of capital are bought on the market. He therefore admits that the prices previously calculated must be adjusted' [42]. Garegnani [27] points out that Marx did not have the method of simultaneous equations available to him and therefore had no other recourse but to adopt a two-step transformation procedure. Duménil complains that 'to pretend the exposition of the transformation in Volume III is perfect does not do service to the memory of a genius thinker' [13]. The transformation procedure, like the whole of Volume III of *Capital*, is unfinished. Marx did not fully resolve the classical contradiction between the law of value and the law of uniform profits. This loophole in Marx's theory of value was subsequently exploited by his critics.

7.2. **The modern context of the transformation problem.** Marx's theory of value was fated to attract a great deal of critical attention, not least due to the radical political conclusions that follow. One of the more important early critics was Bortkiewicz ([80], 1907) who employed simultaneous equations in an effort to complete Marx's transformation. Bortkiewicz assumed a state of self-replacing simple reproduction, thereby abstracting from the historical process of the formation of the general rate of profit, and imposed the constraint that input prices equal output prices. As Bortkiewicz noted, 'Insofar as it is a question of demonstrating Marx's errors it is quite unobjectionable to work with limiting assumptions of this kind, since what does not hold in the special case cannot claim general validity' [80]. Bortkiewicz demonstrated, in a simple model of price-value divergence, that Marx's aggregate conservation rules cannot simultaneously hold. Bortkiewicz's result turned Marx's incomplete transformation into a transformation 'problem' [72] thereby initiating the modern period of the analysis of the classical contradiction, characterised by the use of simultaneous equations to prove the impossibility of Marx's transformation and by implication the logical untenability of his theory of value.

The transformation problem has generated a large literature, of increasing mathematical sophistication, both for and against Marx's labour theory of value. The most influential critical strand employs the Sraffa and Leontief input-output framework (Bortkiewicz was a teacher of the young Leontief [67]). The literature on the transformation problem is very sparse between 1958 and 1971 until Samuelson's Sraffa-inspired critique of Marx's theory of value in the *Journal of Economic Literature* ([31], ch. 14). Subsequently, neo-Ricardian followers of Sraffa, in particular, have repeatedly demonstrated the impossibility of Marx's conservation laws once input and output prices are simultaneously transformed (e.g., Pasinetti [56], Lippi [42], Steedman [77], Abraham-Frois and Berrebi [1], Bidard [5], and many more). Although as Trigg notes 'it has been generally recognised that Marx's transformation procedure represented an important step in the development of value theory' ([79], p.94) the overall consensus is that Marx's theory of value fails, in large part due to the impossibility of reconciling the dual accounting systems of value and price.

Let's examine the neo-Ricardian analysis of the transformation problem, following Pasinetti ([56], appendix to Ch. 5) and Abraham-Frois and Berrebi ([1], Ch. 6). The neo-Ricardian formulation of the transformation problem normally begins with a definition of labour values appropriate for a n-sector Sraffian system.

Definition 7.1. The $1 \times n$ vector $\mathbf{v}$ of *labour values* (or labour-costs) is given by

$$\mathbf{v} = \mathbf{v}\mathbf{A} + \mathbf{l} \tag{35}$$

$$= \mathbf{l}(\mathbf{I} - \mathbf{A})^{-1} \tag{36}$$

$$= \mathbf{l}\mathbf{L}. \tag{37}$$

Recall that $\mathbf{L} = (\mathbf{I}-\mathbf{A})^{-1}$ is the Leontief inverse that represents the total direct and indirect requirements for the production of unit commodities. Hence $\mathbf{lL}$ represents the vertically integrated labour coefficients, or quantities of labour 'embodied' in the corresponding commodities [56]. Sraffa implicitly employs this method of calculating labour values in PCMC via his 'reduction to dated quantities of labour' ([75], ch. VI). Sraffa's iterative reduction, in which costs of a commodity 'resolve themselves' into wages and profits, via vertical integration over the cost structure, is an informal deduction of the matrix power-series representation of prices of production. Sraffa provides a 'reduction equation' ([75], p.35) that in our context is simply the expansion of price solution (3)

$$\mathbf{p} = \mathbf{l}[\mathbf{I} + \mathbf{A}(1+r) + \mathbf{A}^2(1+r)^2 + \ldots]w(1+r). \tag{38}$$

Each term in the expansion is one 'year' earlier in the production process. The quantity of labour embodied in a commodity is then 'the sum of a series of terms when we trace back the successive stages of the production of the commodity' ([75], p.89). Sraffa explains that when the whole net product distributed as wages, that is zero profits, 'the relative values of commodities are in proportion to their labour cost, that is to say to the quantity of labour which directly and indirectly has gone to produce them. At no other wage-level do values follow a simple rule' ([75], p.12). Putting $r = 0$ into equation (38) yields

$$\begin{aligned} \mathbf{p} &= \mathbf{l}[\mathbf{I} + \mathbf{A} + \mathbf{A}^2 + \ldots]w \\ \mathbf{p} &= \mathbf{l}(\mathbf{I} - \mathbf{A})^{-1}w \\ &\propto \mathbf{v}. \end{aligned}$$

Samuelson presents the same definition in his 1971 Sraffa-inspired critique of Marx's theory of value and explains that 'the accuracy of this result can be verified by going back in time to add up the dead labour needed at *all* the previous stages' [67].

The next step is to construct definitions within the formalism that correspond, as close as possible, to Marx's theory of labour-value accounting.

Definition 7.2. The *labour value of constant capital* is

$$C_l = \mathbf{vAq}^{\mathrm{T}}.$$

Definition 7.3. The *labour value of variable capital* is

$$V_l = \mathbf{v}\bar{\mathbf{w}}^{\mathrm{T}}\mathbf{lq}^{\mathrm{T}}.$$

Definition 7.4. *Surplus-value* is

$$S_l = \mathbf{v}(\mathbf{I} - \mathbf{A} - \bar{\mathbf{w}}^{\mathrm{T}}\mathbf{l})\mathbf{q}^{\mathrm{T}}.$$

Next define the corresponding price accounts, where prices $\mathbf{p}$ are defined by Sraffian price equation (2) (or equivalently by equation (8)).

Definition 7.5. The *price of constant capital* is

$$C_p = \mathbf{pAq}^{\mathrm{T}}.$$

Definition 7.6. The *price of variable capital* is

$$V_p = \mathbf{p}\bar{\mathbf{w}}^{\mathrm{T}}\mathbf{l}\mathbf{q}^{\mathrm{T}}.$$

Definition 7.7. *Total profit* is

$$S_p = \mathbf{p}(\mathbf{I} - \mathbf{A} - \bar{\mathbf{w}}^{\mathrm{T}}\mathbf{l})\mathbf{q}^{\mathrm{T}}.$$

Marx's conservation thesis between values and prices can now be formulated.

Definition 7.8. Marx's *conservation thesis* is: (i) conservation of total value in total price,

$$\mu(C_l + V_l + S_l) = C_p + V_p + S_p, \tag{39}$$

(ii) conservation of total surplus-value in total profit,

$$\mu S_l = S_p \tag{40}$$

and (iii) equality of the value rate of profit and the price rate of profit

$$\frac{S_l}{C_l + V_l} = \frac{S_p}{C_p + V_p}, \tag{41}$$

where μ is the MELT. Note that any two of the conservation rules implies the remaining rule.

The impossibility of Marx's conservation thesis then follows.

Theorem 7.1 (The transformation problem)**.** In general, Marx's conservation rules cannot be simultaneously satisfied. Either there is conservation of total value in total price, or there is conservation of total surplus-value in total profit, but not both.

Proof. The first point to note is that values

$$\mathbf{v} = \mathbf{v}\mathbf{A} + \mathbf{l}$$

and prices

$$\mathbf{p} = \mathbf{p}\mathbf{A}^{+}(1 + r)$$

are two distinct systems of evaluation. To relate them we need a MELT, μ. The MELT is implicitly defined by conservation rule $\mu(C_l + V_l + S_l) = C_p + V_p + S_p$ or conservation rule $\mu S_l = S_p$. Therefore, one conservation rule must be taken as true by assumption. For simultaneous satisfaction of both rules the MELTs defined by each rule must be consistent.

Assume $\mu(C_l + V_l + S_l) = C_p + V_p + S_p$; that is, $\mu\mathbf{v}\mathbf{q}^{\mathrm{T}} = \mathbf{p}\mathbf{q}^{\mathrm{T}}$. The conservation rule $\mu S_l = S_p$ can be written as $\mu(\mathbf{v}\mathbf{q}^{\mathrm{T}} - \mathbf{v}\mathbf{A}^{+}\mathbf{q}^{\mathrm{T}}) = \mathbf{p}\mathbf{q}^{\mathrm{T}} - \mathbf{p}\mathbf{A}^{+}\mathbf{q}^{\mathrm{T}}$. Replacing $\mathbf{p}\mathbf{q}^{\mathrm{T}}$ by $\mu\mathbf{v}\mathbf{q}^{\mathrm{T}}$ gives $\mu\mathbf{v}\mathbf{A}^{+}\mathbf{q}^{\mathrm{T}} = \mathbf{p}\mathbf{A}^{+}\mathbf{q}^{\mathrm{T}}$. But $\mathbf{p}\mathbf{A}^{+} = (1/(1 + r))\mathbf{p}$. Therefore, the set of cases for which Marx's thesis holds is defined by the condition,

$$\mathbf{v}\mathbf{q}^{\mathrm{T}} = \mathbf{v}\mathbf{A}^{+}(1 + r)\mathbf{q}^{\mathrm{T}}. \tag{42}$$

But in general *there is no reason* why this condition should hold; hence, the conclusion follows. (Note also that if any one of Marx's conservation claims is assumed to hold, then unless condition (42) is satisfied, at least one of the remaining two claims is false). □

Rewrite equation (42) as

$$\begin{aligned}\mathbf{v}(\mathbf{I}-(1+r)\mathbf{A}^{+})\mathbf{q}^{\mathrm{T}} &= 0\\ \mathbf{x}\mathbf{q}^{\mathrm{T}} &= 0,\end{aligned}$$

where $\mathbf{x}=\mathbf{v}(\mathbf{I}-(1+r)\mathbf{A}^{+})$. Then the consistency condition is the orthogonality of vectors $\mathbf{q}$ and $\mathbf{x}$. Hence, the compositions of gross output that just happen to satisfy Marx's thesis are $\mathbf{q}\in A=\{\mathbf{q}|\mathbf{x}\mathbf{q}^{\mathrm{T}}=0\wedge\mathbf{q}(\mathbf{I}-[\mathbf{A}^{+}]^{\mathrm{T}})\geq\mathbf{0}\}$. Again, there is no economic reason why $\mathbf{q}$ should be a member of set A. However, some subsets of A in which Marx's thesis does hold have an economic interpretation [56, 1].

(1) *Zero profits*, or the case of simple commodity production. If $r=0$ equation (2) is $\mathbf{p}=\mathbf{p}\mathbf{A}+\mathbf{l}w$; hence $\mathbf{p}=\mathbf{l}(\mathbf{I}-\mathbf{A})^{-1}w=\mathbf{v}w$, and prices are proportional to labour values. Condition (42) reduces to the identity $\mathbf{p}\mathbf{q}^{\mathrm{T}}=\mathbf{p}\mathbf{q}^{\mathrm{T}}$.
(2) *Uniform organic composition of capital.* If $\mathbf{l}$ and $\bar{\mathbf{w}}$ are such that $\mathbf{v}\propto\mathbf{v}\mathbf{A}^{+}$ then the economy is operating with uniform organic composition of capital (see [1], p. 158–159). In which case $\mathbf{v}=\mathbf{v}\mathbf{A}^{+}(1+r)$, prices are proportional to labour values and condition (42) reduces to $\mathbf{v}\mathbf{q}^{\mathrm{T}}=\mathbf{v}\mathbf{q}^{\mathrm{T}}$,
(3) *Production in 'standard proportions'.* If $\mathbf{q}=\mathbf{q}[\mathbf{A}^{+}]^{\mathrm{T}}(1+r)$, that is the composition of total output is identical to the composition of the aggregate of commodities, both means of production and wage goods, that must be advanced to obtain that output [42, 67], then condition (42) reduces to the identity $\mathbf{v}\mathbf{q}^{\mathrm{T}}=\mathbf{v}\mathbf{q}^{\mathrm{T}}$.

Marx knew there was no contradiction between the law of value and uniform profits if the organic composition of capital is uniform or profits are zero; indeed, these are the cases when no transformation is required. The special case of 'standard proportions' was not known to Marx and relates to Sraffa's construction of a standard commodity in PCMC.

Pasinetti observes that value accounting, represented by equation (35), and price accounting, represented by equation (2), are 'two completely different systems of equations. It is hardly surprising, then, that they should have different solutions' [56]. The informational gap between values and prices of production prompts Samuelson to exclaim, 'Contemplate two alternative and discordant systems. Write down one. Now transform by taking an eraser and rubbing it out. Then fill in the other one! *Voila!* You have completed your transformation algorithm' [67]. Foley notes that the correctness of mathematical arguments equivalent to theorem 7.1 is 'unquestioned'. In general, labour values and prices are unrelated, unless consistency condition (42) is imposed.

Critics of Marx's theory of value interpret theorem 7.1 to 'show that there is no rigorous quantitative connection between the labour time accounts arising from embodied labour coefficients and the phenomenal world of money price accounts' [20]. For example, Steedman's highly influential critique of Marx's value theory, *Marx After Sraffa* [77], rejects Marx's value theory on two major grounds. First, Marx's value theory is *internally inconsistent* because Marx 'assumes that $S/(C+V)$ is the rate of profit but then derives the result that prices diverge from values, which means precisely, in general, that $S/(C+V)$ is not the rate of profit' ([77], p.31). Steedman concludes that this criticism 'is quite independent of the question whether or not input prices should be transformed. Even more important to notice is the fact that adherents to Marx's 'solution' never attempt a *direct reply* to the above criticism. The reason for this is simple: the criticism is sound and cannot be answered'. Second, Marx's value theory is *redundant* because 'profits and prices *cannot* be derived from the ordinary value schema, that $S/(C+V)$ is *not* the rate of profit and that total profit is *not* equal to surplus value' ([77], p.48). Steedman reiterates Samuelson's eraser critique: given a technique and a real wage (the 'physical schema') one can determine (a) profits and prices and (b) labour-values. But due to the non-satisfaction of condition (42) there is in general 'no way' of relating (a) and (b). *Contra* Marx, labour-value accounting *cannot account* for the phenomenal world of money price accounts. Steedman issues a challenge: 'This conclusion, it should perhaps be emphasized, is the conclusion of an argument of logic; should anyone wish to challenge it, they must do so by either by finding a logical flaw in the argument or by rejecting explicitly and coherently one or more of the assumptions on which it is based' ([77],p.49). Steedman warns that this 'type of argument has been examined, in various forms, by many different writers over the last 80 years. The same conclusions have always been reached and no logical flaw has ever been found in such arguments' [77]. In consequence, '*there is no problem of transforming values into prices, etc., to be solved.* The "transformation problem" is a "non-problem", a spurious problem which can only be thought to arise and to have significance when one is under the misapprehension that the rate of profit must be determined in terms of labour quantities. Once it is seen that there is no such necessity, the "problem" simply evaporates' ([77], p.53).

In sum, the neo-Ricardian critique concludes that labour value accounting fails as an account of exploitation and as an account of the formation of the general rate of profit. The computation of labour values is a complicating detour, best put aside for 'it can scarcely be over-emphasised that the project of providing a materialist account of capitalist societies is dependent on Marx's value magnitude analysis *only* in the negative sense that continued adherence to the latter is a major fetter on the development of the former' ([77], p. 207). The century long history of the transformation problem is a continual reiteration that Marx's transformation from values to prices is 'impossible' [5]. The modern verdict is that Marx's conservation law is false and his theory of value logically flawed.

7.3. **Responses to the transformation problem.** The transformation problem is a logico-mathematical critique of Marx's value theory. The advantage of a formal argument is that the deductive steps normally cannot be contested. Steedman warns that only a restricted set of counter-arguments are possible: either reject an assumption or find a deductive flaw. Yet the possibility of discovering a deductive flaw is unlikely given the number of researchers who have actively studied it. Another response is to accept the argument but provide a different interpretation of its significance for Marx's theoretical project. Let's briefly examine some notable responses to the transformation problem. The examination is indicative not exhaustive.

To my knowledge Sraffa did not present a definitive judgement on the status of the labour theory of value in his published writings, unlike the majority of his interpreters who explicitly reject it. The following extracts from Sraffa's unpublished papers, quoted by Bellofiore [2], are very interesting in this context. Sraffa writes:

> The propositions of M.[arx] are based on the assumption that the comp.[osition] of any large aggr.[egate] of commodities (wages, profits, const[ant] cap.[ital]) consists of a random selection, so that the ratio between their aggr.[egate] (rate of s.[urplus]v.[alue], rate of p.[rofits]) is approx.[imately] the same whether measured at 'values' or at the p.[rices] of prod.[uction] corresp.[onding] to any rate of s.[urplus]v.[alue].
> Quoted in [2], (SP D3/12/111: 140; original in English)

Sraffa adopts a typically Ricardian view of the relation between labour-value and price: it is approximate. The divergences between labour-value and price accounting due to the non-uniformity of the organic compositions of capital 'washes out' in the aggregrate. But Sraffa is aware that this response is open to objection.

> This is obviously true, and one would leave it at that, if it were not for the tiresome objector, who relies on hypothetical deviations: suppose, he says, that the capitalists changed the comp.[osition] of their consumption (of the same aggr.[egate] price) to commod[itie]s of a higher org.[anic] comp.[osition], the rate of s[urplus].v.[alue] would decrease if calc.[ulated] at 'values', while it would remain unchanged at p.[rices] of prod.[uction] which is correct? - and many similar puzzles can be invented.
>
> (Better: the cap[italist]s switched part of their consumption from comm[oditie]s of lower to higher org.[anic] comp.[osition], while the workers switched to the same extent theirs from higher to lower, the aggr.[egate] price of each remaining unchanged...)
>
> It is clear that M[arx]'s pro[position]s are not intended to deal with such deviations. They are based on the assumption (justified in general) that the aggregates *are* of some average composition. This is in general justified in fact, and since it is not intended to be applied to detailed minute differences it is all right.
>
> This should be good enough till the tiresome objector arises. If then one must define which is the average to which the comp.[osite] should conform for the result to be exact and not only approximate, it is the St.[andard] Comm.[odity]...
>
> But what does this average 'approximate' to? i.e. what would it have to be composed of (what weights sh[oul]d the average have) to be exactly the St.[andard] Com.[modity]?

> i.e. Marx *assumes* that wages and profits consist *approximately* of quantities of [the] st.[andard] com.[modity].
>
> Quoted in [2], (SP D3/12/111: 140; original in English)

Sraffa's standard commodity, which has been mentioned in passing, is a composite commodity that has the property that its price is invariant to changes in the distribution of income. A composite commodity that has this property may then serve as an invariable measure of value over all possible nominal wage-profit trade-offs for a given technique and direct labour coefficients. The standard commodity fulfils the role of a real-cost measure in this specific context, in contrast to Marx's labour values that, due to the existence of the transformation problem, cannot perform this function. Sraffa seems to suggest that Marx's value-theory propositions are approximately justified once reformulated in terms of units of the standard commodity, rather than units of labour-time. The problem, of course, is that Marx's conservation claims are exact, not approximate, and are emphatically formulated in units of socially necessary labour-time, not quantities of a composite commodity. This is because, for Marx, price is a necessary form of appearance of labour-time and hence, under certain abstract conditions, there is quantitative identity between labour-time and money values. Also, Marx's labour-values are intended to serve as measures of real-cost over changes in income distribution and technical change, yet it is well-known that the standard commodity is not an invariable measure of value under conditions of technical change. Sinha [71] surveys some of the neo-Ricardian attempts to transpose Marx's concept of exploitation into the Sraffian framework and concludes that the standard commodity cannot solve Marx's transformation problem.

Morishima (1973) [49] accepts the neo-Ricardian critique but proposes that the rate of profit is positive if and only if the rate of surplus-value is positive. Morishima labels this relatively trivial result the 'Fundamental Marxian Theorem' [50]. Exploitation is necessary for profit although strict conservation of labour-time in price does not obtain. Morshima, concludes, however, that the labour theory of value must be abandoned. This rejectionist position is taken by many authors, including Lippi (1979) [42], Steedman (1981) [77] and Roemer (1982) [63], all of whom contend that many of Marx's economics insights remain even if his erroneous value theory is rejected. But all three volumes of *Capital* emphasise that the phenomenal value-forms of capitalist society – price, profit, rent etc. – are nominal representations of labour-time. Marx without a labour theory of value is like a clock without a spring. This type of response to the transformation problem ultimately rejects Marx's economic theory.

Shaikh (1977) [68] iteratively extends Marx's two-step transformation procedure while keeping the sum of prices constant. The algorithm has the property that Marx's labour values converge to prices of production. However, the price rate of profit does not equal the value rate of profit and the total profit does not equal the total surplus-value. Also, any initial price vector converges to prices of production [31], so the significance of starting the iteration with prices proportional to labour-values begs the question. Shaikh [69] contends that although there has to be 'relative autonomy' between price and labour-value accounting, nonetheless the accounting systems are approximately close with empirical differences that are typically small, a position close to Ricardo's suggestion that labour-value is a reasonable approximation to price.

Foley (1982) [18] and Duménil (1983) [13] define the MELT as the ratio of the price of the net product to the amount of living labour expended; that is, $\mu = \mathbf{pn}^{\mathrm{T}}/\mathbf{lq}^{\mathrm{T}}$. According to Foley the 'labour theory of value can be stated simply as the principle that the source of the value added to the mass of commodities produced is the labor expended in producing them' ([19], p.14). The MELT, so defined, directly links the expenditure of living labour to the money value added to the inputs to production. In addition, Foley and Duménil define the value of labour-power as the ratio of the money wage, w, to the MELT, μ, which differs from Marx's definition of the value of labour-power as the value of the real wage. Hence, labour-values are dependent on price, which seems contrary to Marx's aims. Foley argues that an advantage of these definitions is that conservation of total surplus-value in total profit is upheld. Yet the conservation of constant capital and the equality of the value and price rate of profit is not. Hence, Foley and Duménil's interpretation of the labour theory of value implicitly rejects Marx's conservation claims.

Both Shaikh's iterative and Foley and Duménil's definitional solutions are examples of a widespread type of response to the neo-Ricardian critique. A particular set of invariance conditions and definitions are constructed that link the dual accounting systems. Yet Marx's conservation thesis does not hold in general whatever invariance condition is chosen. So this type of response is always accompanied by an explanation of why some of Marx's conservation claims must fail. For example, Shaikh explains the mismatch between surplus-value and profit in terms of transfers out of the 'circuit of capital' into the 'circuit of revenue' [69]. Foley suggests that the mismatch between the value and price rate of profit is due to 'the difference between the labor embodied in the means of production and the labor-time equivalent of the money spent on the means of production using the monetary expression of labor' [20]. But whatever the merits of these explanations the fact remains that labour-value accounting cannot provide a complete quantitative account of price phenomena so the 'logical priority of values over prices has evaporated' [31].

The 'Temporal Single System' (TSS) approach to the transformation problem explictly rejects Bortkiewicz's condition that input prices equal output prices in a system of simultaneous determination (e.g., [22] and the collections [43, 24, 25]) on the grounds that Marx's theory is irreducibly dynamic and concerned with causal change. It is completely unrealistic to assume that an economy reproduces in exactly the same state, period after period, and Marx did not make such an assumption. Any model based on this false assumption will necessarily lead to error [23]. As Marx is not an equilibrium economic thinker his value theory must be formulated in a dynamic and non-equilibrium framework. TSS authors therefore construct sequential models of the transformation in which the process of production necessarily upsets price equilibrium. A great strength of the TSS approach is its rediscovery of the dynamic character of Marx's value theory. But the rejection of equilibrium models throws the baby out with the bathwater. Also, the neo-Ricardian critique is not answered, it is avoided, by rejecting the conditions of the problem. But the reasons for rejecting the conditions are, in my view, not convincing. For example, Bortkiewicz's stipulation that inputs prices equal output prices does have textual support in Marx's work and, *prima facie*, defines simple or 'laboratory' conditions under which one would expect Marx's value theory to hold.

Farjoun and Machover (1989) [17], in their original and groundbreaking work, *Laws of Chaos* [17], develop a probabilistic approach to political economy. They accept the mathematical correctness of the neo-Ricardian critique but explicitly reject the assumption of a uniform rate of profit. They argue that an essential property of capitalist competition is that it continually disturbs the formation of a uniform rate of profit, for 'under any reasonable theorization of the concept of competition, forces that tend to scramble rates of profit away from equality are at least as real and powerful as those that pull towards uniformity' [17]. The capitalist economy is a very large system with a huge number of degrees of freedom; hence, the techniques of statistical, rather than classical, mechanics are required to understand the relationship between labour-value and price. On this basis, Farjoun and Machover develop a non-deterministic theory of the relationship between labour-value and market prices, in which the MELT is a random variable. They conclude that market prices are closely correlated with labour-values. So Marx's Volume I assumption of price-value proportionality is correct in a probabilistic sense. Farjoun and Machover reject the theoretical aim of linking labour-value to 'ideal' prices of production on the grounds that such prices never exist. Clearly this is a complete break with the traditional paradigm of the transformation problem. The advantage of statistical mechanical approaches to political economy, compared to static linear algebra models, are overwhelming, particularly the concept of a *statistical equilibrium* (e.g., see [83]). However, Farjoun and Machover do not explain why we should expect Marx's conservation thesis to fail in the special case of uniform returns to capital invested, even if such a case is ideal or theoretical.

The common property of all these responses to the transformation problem is that they concede that the neo-Ricardian critique is essentially correct on its own terms. The authors differ on the relevance of the critique for Marx's theoretical project.

The neo-Ricardian critique of Marx's theory of value takes direct inspiration from Sraffa's system. Yet Sraffa's representation of simple reproduction obscures the value-theoretic principle of real-cost. When this principle is applied the transformation problem is solved. The unique property of this solution, compared to previous responses, is the rejection of the neo-Ricardian critique *on its own terms*. I demonstrate that Marx's conservation *thesis* is in fact a *theorem* in Sraffa's system. At root the transformation problem is a real-cost accounting error due to the omission of the labour-value of money-capital.

8. Solution of the transformation problem

In section 5.1 two interpretations of real-cost in the circular flow in a state of self-replacing equilibrium were given: the vertical integration interpretation and the market exchange interpretation. Real-cost has another interpretation, which is natural in cases of single production, but less so in joint production. The real-cost of i in terms of j represents the amount of j 'embodied' in a unit of i. In this interpretation we imagine that a unit output of commodity-type j undergoes a series of productive transformations; for example, it gets combined with other commodities to produce outputs k and l. But although j has been used-up to produce commodity-types k and l, and undergone a change in material form, it nonetheless is 'embodied' in k and l. Ricardo and Marx (at least in translation) often use the term 'labour-embodied' to denote their labour-cost theories of value. The labour value vector, $\mathbf{v}$, is interpreted as a formalisation of the classical concept of labour-embodied (e.g., [56]). Marx's labour values, in the case of self-replacing equilibrium, are real-costs measured in units of labour. So we can apply lemma 5.4, which defines real-costs measured in terms of any commodity, labour or otherwise, to derive an expression for labour values in the circular flow.

The important point to immediately note is that Sraffa's definition of labour values $\mathbf{v}$ (equation (35)) *differs* from the labour values $\tilde{\mathbf{v}}$ derived from the principle of real-cost (equation (43) below). The following lemma derives the expression for real-cost labour values $\tilde{\mathbf{v}}$. I will then explain the reason for the discrepancy $\mathbf{v} \neq \tilde{\mathbf{v}}$.

Lemma 8.1. *Labour values* in the circular fow are

$$\tilde{\mathbf{v}} = \mathbf{l}(\mathbf{I} - \tilde{\mathbf{A}})^{-1}, \tag{43}$$

where $\tilde{\mathbf{A}}$ is the *technique augmented by capitalist consumption*,

$$\tilde{\mathbf{A}} = \mathbf{A} + \bar{\mathbf{c}}^{\mathrm{T}}\bar{\mathbf{m}}. \tag{44}$$

(Recall that $\mathbf{l}$ is the $1 \times n$ vector of direct labour coefficients, $\mathbf{A}$ is the $n \times n$ technique, $\bar{\mathbf{c}}$ is the $1 \times n$ vector of capitalist consumption coefficients, and $\bar{\mathbf{m}}$ is the $1 \times n$ vector of unit costs.)

Proof. Labour values are the vector of real-costs, $\mathbf{k}_{n+1}$, measured in terms of labour-cost. Two labour costs are of especial interest: the labour-cost of labour, $k_{n+1,n+1} = 1$ by definition 5.6, and the labour-cost of money-capital, $k_{n+2,n+1} = \omega$, measured in units of labour per currency unit supplied. By lemma 5.4,

$$\begin{aligned}
\mathbf{k}_{n+1}[n+1] &= \mathbf{k}_{n+1}[n+1] \begin{bmatrix} \mathbf{A} & \bar{\mathbf{c}}^{\mathrm{T}} \\ \bar{\mathbf{m}} & 0 \end{bmatrix} + \begin{bmatrix} \mathbf{l} & 0 \end{bmatrix} && (45) \\
&= \begin{bmatrix} \mathbf{k}_{n+1}[n+1,n+2]\mathbf{A} + \omega\bar{\mathbf{m}} & \mathbf{k}_{n+1}[n+1,n+2]\bar{\mathbf{c}}^{\mathrm{T}} \end{bmatrix} + \begin{bmatrix} \mathbf{l} & 0 \end{bmatrix}. && (46)
\end{aligned}$$

From (46) we get two equations,

$$\begin{aligned}
\tilde{\mathbf{v}} &= \tilde{\mathbf{v}}\mathbf{A} + \omega\bar{\mathbf{m}} + \mathbf{l} && (47) \\
\omega &= \tilde{\mathbf{v}}\bar{\mathbf{c}}^{\mathrm{T}}, && (48)
\end{aligned}$$

where $\tilde{\mathbf{v}} = \mathbf{k}_{n+1}[n+1, n+2]$ is a vector of labour values excluding the value of labour and money-capital.

Equation (47), in contrast to the Sraffian equation (35) for labour values, has an extra term, $\omega\bar{\mathbf{m}}$, which counts the labour-cost of the money-capital supplied to production. The labour value of money-capital, ω, defined by equation (48), is the amount of labour used-up per unit of money-capital supplied, which is equal to the labour value of capitalist consumption, $\tilde{\mathbf{v}}\bar{\mathbf{c}}^{\mathrm{T}}$.

Substitute (48) into (47) to get

$$\tilde{\mathbf{v}} = \tilde{\mathbf{v}}(\mathbf{A} + \bar{\mathbf{c}}^{\mathrm{T}}\bar{\mathbf{m}}) + \mathbf{l}.$$

Let $\tilde{\mathbf{A}} = \mathbf{A} + \bar{\mathbf{c}}^{\mathrm{T}}\mathbf{m}$ and rearrange to yield

$$\tilde{\mathbf{v}} = \mathbf{l}(\mathbf{I} - \tilde{\mathbf{A}})^{-1}$$

as required. □

The following corollary derives the Sraffian definition of labour value from the principle of real-cost. The important result is that Sraffian values measure labour-cost only in the special case of a circular flow without a capitalist class, that is simple commodity production.

Corollary 8.2. *Sraffian labour values* are the labour values of commodities produced under conditions of simple commodity production,

$$\mathbf{v} = \mathbf{l}(\mathbf{I} - \mathbf{A})^{-1}. \tag{49}$$

Proof. A capitalist class is absent in simple commodity production, that is $r = 0$ and $\bar{\mathbf{m}} = \bar{\mathbf{c}} = \mathbf{0}$. Hence, by lemma 8.1, $\tilde{\mathbf{A}} = \mathbf{A}$ and the conclusion follows. □

The discrepancy $\mathbf{v} \neq \tilde{\mathbf{v}}$ is now easy to understand. *Sraffian labour values do not count the labour-time worked for the production of capitalist consumption.*

In general, $\tilde{\mathbf{v}} > \mathbf{v}$ because $\tilde{\mathbf{v}}$ counts the labour embodied in money-capital, or, equivalently, the labour used-up per unit of money-capital supplied. This labour-time is not apparent in Sraffa's system because money-capital is not viewed as a commodity with an associated real-cost but instead is viewed merely as a nominal parameter that controls distribution. But just as the value of labour-power is the value of the real wage the value of money-capital is the value of capitalist consumption. In classical terminology the money-capital supplied to production is an 'embodiment' of labour that gets 'transferred' to the product, just like any other means of production. More plainly, labour-time is expended to produce the goods that capitalists consume simultaneously with the advancement of money-capital. This labour is missing in Sraffian labour value accounting.

The results that link labour-value in the circular flow to labour-value in the Sraffian system have already been obtained. A special case of the principle of real-cost in the circular flow is the proportionality of prices to labour costs.

Corollary 8.3 (Circular flow prices are proportional to labour cost)**.**

$$\mathbf{p}^{\circ}[n+1] = \mathbf{v}^{\circ}w^{\circ},$$

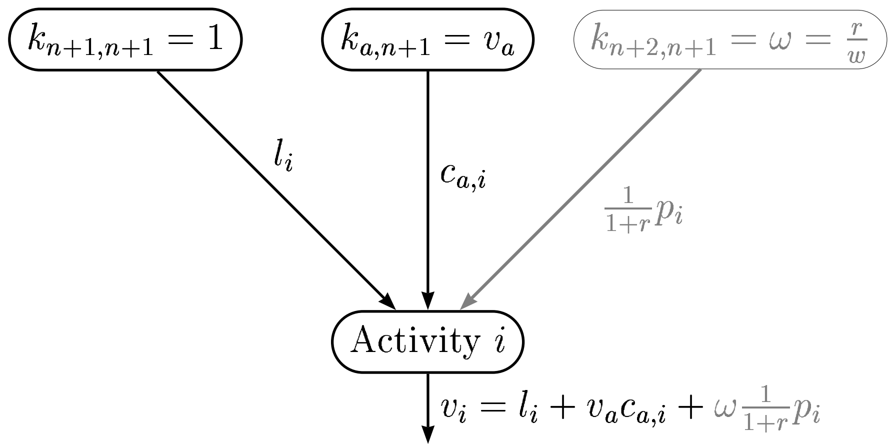


FIGURE 11. **The transformation problem: a real-cost accounting error due to the omission of the labour value of money-capital**. The labour cost of commodity i is the weighted sum of the unit labour cost of all its inputs, in this case the value of labour-power, l_i, the value of a commodity, v_a, and the value of money-capital supplied to production, ω. Neo-Ricardian labour cost accounting omits an input, the value of the money-capital (shown in grey). Hence labour costs are not conserved, except in special cases, such as zero profits ($r = 0$ implies $\omega = 0$). The non-conservation of labour cost manifests as the transformation problem.

where $\mathbf{p}^{\circ}[n+1]$ is a $n+1 \times n+1$ row vector of all prices excluding labour, $\mathbf{v}^{\circ} = \mathbf{k}_{n+1}[n+1]$ is a $n+1 \times n+1$ row vector of labour values, and $w^{\circ} = p^{\circ}_{n+1}$ is the price of labour.

Proof. Labour is the $n+1$th commodity in the circular flow so the conclusion follows by theorem 5.8. □

The general result of prices of production proportional to real-costs implies the particular result of prices of production proportional to labour values; that is, Sraffian prices $\mathbf{p}$ are proportional to real-cost labour values $\tilde{\mathbf{v}}$. (Only in the special case of zero profits are the Sraffian labour values $\mathbf{v}$ proportional to Sraffian prices.)

Proposition 8.1 (Prices of production are proportional to labour values).

$$\mathbf{p} = \tilde{\mathbf{v}} w.$$

Proof. $\mathbf{p}^{\circ}[n+1] = \mathbf{v}^{\circ} w$, by corollary 8.3. But $\mathbf{p}^{\circ}[n+1] = [\mathbf{p}, r]$ by theorem 6.1; hence $[\mathbf{p}, r] = \mathbf{v}^{\circ} w$. Ignoring the price of capital, $\mathbf{p} = \tilde{\mathbf{v}} w$, where $\tilde{\mathbf{v}} = \mathbf{v}^{\circ}[n+1] = (\mathbf{k}_{n+1}[n+1])[n+1] = \mathbf{k}_{n+1}[n+1, n+2]$. □

The belief that prices of production diverge from labour-values has been a canard of political economy since the time of Smith. Marx constructed his transformation based on this assumption. Bortkiewicz retained it, and Sraffians and Marxists have promulgated it. But in a state of self-replacing equilibrium the assumption is false.

Sraffian labour values are *conservative* in the case of simple commodity production, because money-capital is absent, and *non-conservative* in the case of capitalist production, because money-capital is present. One would think, therefore, that this accounting error would have been quickly noticed. But of course it has – in a highly inverted manner. The failure to account for the labour-time embodied in money-capital manifests, in the modern context, as the transformation problem, which is precisely a non-conservation result. An erroneous specification of labour-cost is the cause of the transformation problem. The correct specification of real-cost, of which labour-cost is a special case, is conservative in price (corollary 6.3 and proposition 8.1). This is a consequence of the value-theoretic principle of real-cost, which is hidden in Sraffa's 'surplus' representation that interrupts the circular flow. The proportionality of prices of production to real-costs is therefore missed. The key technical reason is that correct real-cost accounting in the circular flow requires the j-inverse (see definition 5.8) not the Leontief inverse. The vertical integration represented by the Sraffian equation for labour values, $\mathbf{v} = \mathbf{l}(\mathbf{I} - \mathbf{A})^{-1}$, fails to integrate the labour cost of capitalist consumption. Figure 11 depicts the accounting error and appendix D contains a simple numerical example that demonstrates the error.

In consequence, a real-cost accounting error is at the basis of all the value-theoretic conclusions of the neo-Ricardian school, including the proof of the impossibility of Marx's conservation thesis. Once correct real-cost accounting is adopted then the solution of the transformation problem follows, and the apparent contradiction between the law of value and the law of equal returns to capital invested is resolved. Prices of production are proportional to labour values, there is no price-value divergence, and hence all Marx's aggregate equalities hold.

Theorem 8.4 (Marx's conservation theorem). (i) Total value is conserved in total price,

$$\mu(C_l + V_l + S_l) = C_p + V_p + S_p, \tag{50}$$

(ii) total surplus-value is conserved in total profit,

$$\mu S_l = S_p \tag{51}$$

and (iii) the value rate of profit equals the price rate of profit

$$r_v = \frac{S_l}{C_l + V_l} = \frac{S_p}{C_p + V_p} = r, \tag{52}$$

where r_v is the value rate of profit and $\mu = w$ (the wage rate) is the MELT.

Proof. Conservation rule (i):

$$\begin{aligned} \mu(C_l + V_l + S_l) &= C_p + V_p + S_p \\ \mu\tilde{\mathbf{v}}\mathbf{q}^{\mathrm{T}} &= \mathbf{p}\mathbf{q}^{\mathrm{T}} \end{aligned}$$

By proposition 8.1, $\mathbf{p} = \tilde{\mathbf{v}}w$; therefore,

$$\mu\tilde{\mathbf{v}}\mathbf{q}^{\mathrm{T}} = w\tilde{\mathbf{v}}\mathbf{q}^{\mathrm{T}}.$$

Therefore, conservation rule (i) holds with $\mu = w$.

Similarly, conservation rule (ii):

$$\begin{aligned} \mu S_l &= S_p \\ \mu\tilde{\mathbf{v}}(\mathbf{I} - \mathbf{A} - \bar{\mathbf{w}}^{\mathrm{T}}\mathbf{l})\mathbf{q}^{\mathrm{T}} &= \mathbf{p}(\mathbf{I} - \mathbf{A} - \bar{\mathbf{w}}^{\mathrm{T}}\mathbf{l})\mathbf{q}^{\mathrm{T}} \\ \mu\tilde{\mathbf{v}}(\mathbf{I} - \mathbf{A} - \bar{\mathbf{w}}^{\mathrm{T}}\mathbf{l})\mathbf{q}^{\mathrm{T}} &= w\tilde{\mathbf{v}}(\mathbf{I} - \mathbf{A} - \bar{\mathbf{w}}^{\mathrm{T}}\mathbf{l})\mathbf{q}^{\mathrm{T}}. \end{aligned}$$

Therefore, conservation rule (ii) holds with $\mu = w$.

Finally, conservation rule (iii):

$$\begin{aligned} \frac{S_l}{C_l + V_l} &= \frac{\tilde{\mathbf{v}}(\mathbf{I} - \mathbf{A} - \bar{\mathbf{w}}^{\mathrm{T}}\mathbf{l})\mathbf{q}^{\mathrm{T}}}{\tilde{\mathbf{v}}\mathbf{A}\mathbf{q}^{\mathrm{T}} + \tilde{\mathbf{v}}\bar{\mathbf{w}}^{\mathrm{T}}\mathbf{l}\mathbf{q}^{\mathrm{T}}} \\ &= \frac{\mathbf{p}(\mathbf{I} - \mathbf{A} - \bar{\mathbf{w}}^{\mathrm{T}}\mathbf{l})\mathbf{q}^{\mathrm{T}}}{\mathbf{p}\mathbf{A}\mathbf{q}^{\mathrm{T}} + \mathbf{p}\bar{\mathbf{w}}^{\mathrm{T}}\mathbf{l}\mathbf{q}^{\mathrm{T}}} \\ &= \frac{[\mathbf{p} - \mathbf{p}(\mathbf{A} + \bar{\mathbf{w}}^{\mathrm{T}}\mathbf{l})]\mathbf{q}^{\mathrm{T}}}{\mathbf{p}(\mathbf{A} + \bar{\mathbf{w}}^{\mathrm{T}}\mathbf{l})\mathbf{q}^{\mathrm{T}}}. \end{aligned}$$

Therefore, by equation (11),

$$\frac{S_l}{C_l + V_l} = \frac{(\mathbf{p} - \bar{\mathbf{m}})\mathbf{q}^{\mathrm{T}}}{\bar{\mathbf{m}}\mathbf{q}^{\mathrm{T}}}.$$

And by equation (10),

$$\frac{S_p}{C_p + V_p} = \frac{[\bar{\mathbf{m}}(1 + r) - \bar{\mathbf{m}}]\mathbf{q}^{\mathrm{T}}}{\bar{\mathbf{m}}\mathbf{q}^{\mathrm{T}}} = r,$$

where r is the rate of profit or price of money-capital. □

It is a remarkable and deep irony that the modern formulation of the transformation problem employs a definition of labour values that only holds in simple commodity production, that is production absent capitalist exploitation. The literal failure to cost the time worked for the production of capitalist consumption manifests as the transformation problem. The special cases in which Marx's conservation thesis was thought to hold – zero profits, uniform organic composition of capitals, and production in 'standard proportions' – are in fact the special cases in which Sraffian non-conservative labour-cost accounting just so happens to have a proportionate relationship to conservative labour-cost accounting. The neo-Ricardian critics of Marx's theory of value have identified the conditions for which their own real-cost accounting error is accidentally consistent with the general principle of conservation of real-cost in price.

Two issues must be immediately addressed before discussing this theoretical inversion: the apparent link between labour-values and prices, and the real link between labour-values and income distribution.

8.1. **The rate of surplus-value as a distributional variable.** Labour-values $\tilde{\mathbf{v}}$ are a function of unit cost prices $\bar{\mathbf{m}}$,

$$\tilde{\mathbf{v}} = \tilde{\mathbf{v}}\mathbf{A} + \omega\bar{\mathbf{m}} + \mathbf{l}. \tag{53}$$

Equation (53) arises naturally from the independent variables specified by a Sraffian system. But the Sraffian system is by no means the only possible specification of the independent variables of a circular flow. Equation (53) is useful for understanding how the labour-cost of money-capital is essential for correct labour-cost accounting. But it does not entail that labour-values are derivative of price magnitudes. Real-cost and price are closely related and mutually consistent in a state of self-replacing equilibrium. But they are dual accounting systems.

Lemma 8.5. Labour-values are *independent of price magnitudes* and given by

$$\begin{aligned} \tilde{\mathbf{v}} &= (\tilde{\mathbf{v}}\mathbf{A} + \mathbf{l})(1 + r_v) && (54) \\ &= [\mathbf{I} - \mathbf{A}(1 + r_v)]^{-1}\mathbf{l}(1 + r_v), && (55) \end{aligned}$$

where r_v is the value rate of profit, determined by the technique and real-wage, which satisfies Marx's aggregrate labour-value accounting,

$$r_v = \frac{S_l}{C_l + V_l} = \frac{eV_l}{C_l + V_l},$$

where $e = S_l/V_l$ is the rate of surplus-value (or rate of exploitation).

Proof. This follows directly from proposition 8.1 applied to the Sraffian price equation. Alternatively, $\bar{\mathbf{m}} = \mathbf{pA} + \mathbf{l}w = (\tilde{\mathbf{v}}\mathbf{A} + \mathbf{l})w$ by proposition 8.1. Substitute into (53) to yield

$$\tilde{\mathbf{v}} = (\tilde{\mathbf{v}}\mathbf{A} + \mathbf{l})(1 + \omega w).$$

By corollary 5.8, $r = \omega w$. And by Marx's conservation theorem, $r = r_v$. Hence the conclusion follows. $\square$

This equivalent definition of labour-value, $\tilde{\mathbf{v}} = (\tilde{\mathbf{v}}\mathbf{A} + \mathbf{l})(1 + r_v)$, is dependent on the value rate of profit but independent of price. The value rate of profit, r_v, is price independent because it may be calculated from the technique and real-wage via the eigenvalue equation $\tilde{\mathbf{v}} = \tilde{\mathbf{v}}\mathbf{A}^{+}(1 + r_v)$. The value rate of profit plays a symmetrical role in labour-cost accounting to that of the price rate of profit in price accounting. Hence, once the labour-cost distributional variable r_v is fixed then labour-values can be calculated. Alternatively, if the real-wage is fixed then the value rate of profit may be derived. Price accounting need play no role. Indeed, prices may be directly derived from labour-cost accounting by scaling each commodity value by the nominal wage rate.

In Marx's theory the rate of profit must be derived from the rate of surplus-value because the cause of profit is the extraction of surplus-labour during production organised under the social relations of the capitalist firm. Exploitation is the *ex ante* extraction of labour-time from workers in the 'hidden abode of production' [44], rather than a conflict over the *ex post* distribution of a net product in the sphere of circulation. 'The transformation of surplus-value into profit must be deduced from the transformation of the rate of surplus-value into the rate of profit, not vice versa. And in fact it was the rate of profit which was the historical point of departure. Surplus-value and rate of surplus-value are, relatively, the invisible and unknown essence that wants investigating, while rate of profit and therefore the appearance of surplus-value in the form of profit are revealed on the surface of the phenomenon' ([45], p.43). For Marx the rate of profit primarily depends on how much surplus-labour capital succeeds in extracting from the workforce. Competition only effects the equalisation of profits between individual capitals at this predetermined level. This view is importantly different from the neo-Ricardian problem of the distribution of a net product 'after the harvest'. Ricardo studied the economic consequences of different distributions of the net income; Marx critiqued the social consequences of the production of the net income under conditions of capitalist exploitation.

In consequence of lemma 8.5 labour-cost accounting can fully and consistently perform the theoretical role that Marx intended for it. Marx's claim that 'the rate of profit, therefore, depends on two main factors – the rate of surplus-value and the value-composition of capital' ([45],p.69) is confirmed. Of course, in a state of self-replacing equilibrium, represented by sets of simultaneous equations, nothing much can be said about causality. Under conditions of dynamic change, labour and price accounting have important causal dependencies. But the Marxist starting point, which emphasises the rate of exploitation as a key distributional variable, is not only logically consistent but is a more accurate depiction of the locus of profit formation. The independence of labour-value from price was initially hidden only because we began with a critique of the Sraffian system.

8.2. **Labour-value and income distribution.** Sraffian labour-value accounting is independent of income distribution: whatever the rate of profit, r, all other things being equal, labour-values are unchanged. In contrast, labour-values derived from the principle of real-cost are dependent on income distribution.

Direct labour-costs, $\mathbf{l}$, are purely technical facts that represent the direct 'toil and trouble' of producing particular commodities. Labour values, $\tilde{\mathbf{v}}$, however, do depend on the distribution of the net product, because they measure the *total* 'toil and trouble' of producing commodities. Under conditions of capitalist income distribution it is not possible to produce 1 unit of corn, iron etc. without also simultaneously producing a part of the capitalist consumption bundle. This real-cost of money-capital is a socially necessary cost; it must be met for money-capital to be supplied. Labour-costs must measure this additional 'difficulty' of production and are therefore necessarily dependent on income distribution.

Consider a Sraffian system $\Psi_{r=0}$, in which profits are zero (simple commodity production), and a system $\Psi_{r>0}$, in which profits are non-zero (simple reproduction). Assume that both systems are *identical in all respects* except the same net product is $\mathbf{n} = \mathbf{w}$ when profits are zero, and $\mathbf{n} = \mathbf{w} + \mathbf{c}$ when profits are non-zero. For simplicity, assume further that the composition of worker and capitalist consumption is identical, that is $\mathbf{c} = \alpha\mathbf{w}$. Now, $\tilde{\mathbf{v}} = \mathbf{v}$ in $\Psi_{r=0}$ (corollary 8.2); whereas in general

$\tilde{\mathbf{v}} > \mathbf{v}$ in $\Psi_{r>0}$. Labour values are higher in $\Psi_{r>0}$ because *more total labour-time* is needed to produce unit commodities due to the requirement to simultaneously produce the goods that capitalists consume. Labour values are lower in $\Psi_{r=0}$ because *less total labour-time* is needed to produce unit commodities because there is no need to simultaneously produce the goods that capitalists consume. We can think of the transition from simple commodity production to simple reproduction as the addition of a new basic commodity, money-capital, to the technique, which represents a new kind of property claim. Indirectly, the advance or 'production' of money-capital requires labour as an input. Hence, the 'difficulty' of production of all commodities, when measured in terms of labour-time, rises.

In contrast, Sraffian vertical integration over the cost structure omits the real-cost of money-capital. Hence Sraffian labour-values are *identical* in simple commodity production, $\Psi_{r=0}$, and simple reproduction, $\Psi_{r>0}$. The specific character of capitalist production, in which money-capital is an essential input to the productive process, is ignored.

8.3. **The theoretical inversion.** Both Ricardo and Marx thought that prices diverge from labour values due to profit-equalizing rates of return. This premiss of the transformation problem has been universally accepted. But in the modern context of self-replacing equilibrium and simultaneous determination the premiss is mistaken. To assume divergence and attempt to solve the transformation problem is already an error; hence, over one hundred years of negative results, and the disappearance of the importance of real-cost in economics. Marxist political economy has maintained the divergence and the aggregate conservation, whereas neo-Ricardian political economy has maintained the divergence and denied the aggregate conservation. Yet on this point both sides of the argument are wrong: there is no divergence and there is aggregate conservation.

Marx does not distinguish between equilibrium and non-equilibrium states to the level of precision obtained by linear production theory. Furthermore, Marx's analysis is irreducibly dynamic. So it is not possible to capture the full generality and meaning of Marx's theory of value in a linear production framework. For example, in an uninterrupted circular flow no additional surplus-value is created so Marx's concept of *variable* capital, which is precisely that portion of capital that during production does not regularly conserve and transfer labour-cost, necessarily plays a vestigial rather than central role. A further example of the incongruity between Marx's theory and linear production theory is our finding that, in the special case of self-replacing equilibrium, Marx's Vol. I assumption of price-value proportionality extends to the profit-equalising prices of production examined in Vol. III, contrary to what Marx would have expected. Proportionality between prices of production and labour value contradicts the basic tenet of Marx's transformation procedure, that prices of production diverge from labour values. But the Bortkiewiczian 'simultaneous' interpretation of Marx's transformation procedure, although dominant, is one plausible interpretation amongst many. Hence, the extent to which Marx's supposition of price-value divergence retains a role in a dynamic context under conditions of technical change is, to this author, an open question.

Marx improved upon Ricardo's early numerical examples of the classical contradiction, which he justifiably called 'clumsy' [46]. Nonetheless the reasons Marx gives for the existence of price-value divergence are not always convincing, which is to be expected given the results we have obtained. For example, in Ch. 9 of Vol. III, on the transformation, Marx proceeds as by proof by contradiction, but the logic is not quite correct (cf. section 7.1). He initially assumes (i) uniform rates of surplus-value in the different sectors and (ii) that commodities exchange according to value. He then notes that the value rate of profit can only be uniform if sectors have a uniform organic composition of capital. But in general this cannot be. So Marx deduces that *both* his initial assumptions do not hold: (i) commodities do not exchange according to their values (price-value divergence) and (ii) the rates of surplus-value in each sector differ. But Marx does not show that both his initial assumptions generate the contradiction, which would be required to reject them. He does not consider the possibility that only one of his initial assumptions – uniform rates of surplus-value – need be relaxed. It should be stressed that the existence in Marx's theory of price-value divergence due to profit-equalisation is not based on empirical evidence of any kind, but derives solely from the existence of a logical contradiction in the Ricardian theory of value.

Correct labour-cost accounting under conditions of simultaneous determination partially manifests in the neo-Ricardian literature as the reduction to dated quantities of labour. But reduction equation (38) (and various analogues) is incomplete because it leaves the price, r, of money-capital *unreduced*. The rate of profit is the price of money-capital and money-capital is a means of production with an associated labour-cost. Sraffa's *Production of Commodities by Means of Commodities* excludes the special commodity that uniquely differentiates capitalist production from simple commodity production, that is Marx's *Capital*, and hence fails to properly formulate real-cost accounting, resulting in a definition of labour value appropriate only for production absent a capitalist class. This omission pervades all subsequent neo-Ricardian discussions of value theory. In consequence, the inconsistency and redundancy critiques of Marx's theory of value are easily refuted.

The inconsistency critique fails because the value rate of profit is identical to the price rate of profit. Marx's claim that $S/(C+V)$ is the rate of profit is correct. And labour-values magnitudes can be determined without reference to prices.

The redundancy critique fails because profits and prices can be derived from labour-value accounting. Prices can be directly derived from labour-cost accounting by scaling each commodity value by the the nominal wage rate. Labour-value accounting is not a 'complicating detour' but essential for understanding how surplus-labour extracted at the point of production is the source of profit in capitalist economy (which, as Shaikh notes, is immediately apparent once production ceases via strikes).

I conclude that the standard neo-Ricardian critique of Marx's theory of value is refuted. Sraffa's prices of production are proportional to labour-values. The dual accounting systems are mutually consistent, two sides of the same coin. The neo-Ricardian school of political economy therefore has two choices: either accept the consistency and non-redundancy of Marx's theory of value within their own modelling framework, or reject the principle of real cost and with it the possibility of completing the classical approach to the theory of economic value.

Lavosier, in the 18th century, overthrew the theory of phlogiston by carefully weighing the input mass and output mass of chemical reactions. He discovered, making use of the balance, that mass is a property of a substance that is never obliterated. Phlogisticians, in contrast, required auxiliary hypotheses to explain mass transfers during combustion. Lavosier's principle of the conservation of matter laid the foundations of modern quantitative chemistry. Conservation principles have always been fundamental to scientific understanding [47]. The transformation problem can be viewed as the symptom of a failed conservation principle in the domain of economic theory. This has led to the rejection of objective 'substance' theories of value and the concomitant loss of real-cost conservation principles [48]. But the problem is not the idea of a 'substance' theory of economic value. The real problem is the accounting, or balancing, of 'substance' flows. The modern history of the transformation problem is in fact the history of an accounting error. Labour value is conserved in price, under conditions of simultaneous determination and a realised uniform rate of profit, once real-costs are properly accounted for.

9. Implications for the theory of economic value

The principle of real-cost now needs to be applied to linear production models that include fixed-capital and joint production. Many properties of the single-production Sraffian system do not transfer to joint production [5] and some neo-Ricardian criticisms of Marx's theory of value are specific to joint production [77, 78]. An important question is to what extent the negative results depend on the transfer of an interrupted circular flow and undistributed surplus from single-production systems to joint-production systems. The failure to theorise the principle of real-cost in the simpler case of single-production implies that existing joint-production analyses are inadequate.

A full mathematical formulation of Marx's theory of value requires dynamics. A dynamic theory must include changes in absolute and relative surplus-value and trace the causal consequences for capital accumulation. In stark constrast, a state of self-replacing equilibrium exhibits a great deal of symmetry precisely because it represents only mutual and simultaneous consistency rather than causal change. Prices of production are proportional to real-cost measured in terms of any real-cost basis. Marx's theory of value is logically consistent and non-redundant in the special case of self-replacing equilibrium yet there are a surfeit of real-cost theories to choose from. The theory of value is therefore under-determined in this special case. I believe this is necessarily the case on philosophical grounds because value-theory must ultimately be a causal theory (e.g. [82]) and change is precisely what is absent in the Sraffian representation of an economy, in which money can only function as a means of exchange, rather than a 'transmission belt' for the reallocation of social labour time under conditions of technical change. My hope is that this paper will relocate the 'value controversy' from statics to dynamics.

A major difference between the circular flow representation of simple reproduction and Sraffa's surplus equations is the presence of money-capital as a factor of production. In this sense the circular flow is a monetary-production economy. The model therefore needs to be related to the wider literature on analyses of monetary-production [29] and the recent work of Trigg [79] who analyses Marx's Volume II reproduction schemes in the context of Keynes, Kalecki and Sraffa. Marx's theory of commodity-money is not fully integrated with his transformation procedure because he assumed that the labour-value of money is exogenously determined and constant throughout ([45], p.50). In the circular flow we assumed that money is a pure symbol conserved in exchange. Money-capital, which is money supplied to production that receives a return, is a commodity, not because it is metal that requires direct labour for its production, but because it has a price and an associated real-cost. This concept of money as commodity is subtly different from Marx's concept. Marx emphasised that the 'value of money' must be causally constrained by the 'value of the money commodity' if paper money is to function as a symbol of value at all: 'Only in so far as paper money represents gold, which like all other commodities has a value, is it a symbol of value' ([45], p. 129). I think, therefore, that more work is required to fully settle on a Marxist theory of the value of money and money-capital. The concepts of money and money-capital described in this paper should be integrated with the monetary approaches to Marx's transformation problem pioneered by Bellofiore [3], who considers bank credit and Schumpeterian themes, and Moseley [51, 52], who emphasises the causal role of variable capital under conditions of sequential determination.

Engels, in a subsequent *Supplement to Capital, Volume III* [16], claimed that Marx's law of value underwent a modification in the transition from pre-capitalist to capitalist economic formations:

> Marx's law of value applies universally, as much as any economic laws do apply, for the entire period of simple commodity production, ie. up to the time at which this undergoes a modification by the onset of the capitalist form of production.... Thus the Marxian law of value has a universal economic validity for an era lasting from the beginning of the exchange that transforms products into commodities down to the fifteenth century of our epoch... a period of some five to seven millennia. [16]

According to Engels, and arguably Marx, once there are developed markets then mismatches between supply and demand cause prices to diverge from values, but the onset of capitalism introduces an additional cause of divergence, the formation of prices of production, a process that 'modifies' the operation of the law of value.

The meaning of simple commodity production is a subject of debate in Marxist economics. The conclusion of this paper, that prices are proportional to labour-values, both under conditions of simple commodity production and capitalist reproduction, removes one of the key assumptions of the debate. The implications of this result for the logical and historical status of simple commodity production need to be developed.

Marx's conservation theorem has implications for Marxist empirical work that uses Leontief's input-output tables to calculate Sraffian embodied labour coefficients and prices of production for real-world economies (e.g., [70, 54, 7]). The motivation for much of this work is to check whether Marx's labour-value accounting has operational validity despite price-value divergences due to profit-rate equalisation. The main finding is labour-values are closely correlated with both prices of production and market prices [20]. But the interpretation of this data must be re-examined to the extent that the Sraffian, or simple commodity production, definition of labour-value is employed. The task of applying Marxist labour-cost accounting to real-world data may well be simpler and conceptually clearer once conservative labour-cost accounting is adopted.

A sceptical stance toward the possibility of a theory of economic value does not find support in Sraffa's theory, contrary to prevailing intepretations, because implicit within Sraffa's theory is the intuitive and simple value-theoretical principle of real-cost. Sraffa remarks that his results 'cannot be reconciled with *any* notion of capital as a measurable quantity independent of distribution and prices' [75]. But this statement is not complete. In a state of self-replacing equilibrium there are *many* measures of capital independent of prices, of which labour-value is just one example. However, such real-cost measures of capital *are* dependent on the distribution of real income. This means that, at least in the case of Marx's theory of value, Sraffa's critique does not hold. This opens up some new possibilities in the theory of value. For example, Marx's conservation theorem has implications for Ricardo's problem of an 'invariable measure of value'.

9.1. **Towards an invariable measure of value.** The problem of real-cost accounting given a Sraffian system Ψ differs from the problem of comparative real-cost accounting given a pair of Sraffian systems Ψ_1 and Ψ_2. In this paper we have been exclusively concerned with the former problem. The latter problem is closely related to Ricardo's search for an 'invariable measure of value'. Ricardo initially adopted labour-cost as the invariable measure but, as Bidard explains, became 'fully aware of the analytical defects of labour value: (i) the prices are not proportional to labour contents, (ii) for a given quantity of labour, the price depends on the delay of production and (iii) relative prices, but not labour values, are affected by changes in distribution' ([5], p. 60). The transformation problem was the obstacle that prevented Ricardo from using labour-cost as an invariable measure. But, as we have seen, all these conclusions, at least in the case of self-replacing equilibrium, are false. The subject of an invariable measure of value is subtle with far-reaching consequences.

Call $k_{i,i}$, the real-cost of i in terms of i, the *index-cost*. $k_{i,i} = 1$ is true by definition for all Ψ and is independent of changes of technique or income distribution. Given a Sraffian system Ψ we can choose any commodity as a real-cost basis. Let's choose labour. For Marx 'the value of labour-power is determined, as in the case of every other commodity, by the labour-time necessary for the production, and consequently also the reproduction, of this special article. So far as it has value, it represents no more than a definite quantity of the average labour of society incorporated in it' [44]. The value of the real wage is $\tilde{\mathbf{v}}\bar{\mathbf{w}}^{\mathrm{T}}$. As $\mathbf{p}\bar{\mathbf{w}}^{\mathrm{T}} = w$ then by proposition, 8.1 $\tilde{\mathbf{v}}\bar{\mathbf{w}}^{\mathrm{T}} = k_{n+1,n+1} = 1$. This follows by definition 5.6: the real-cost of i in terms of i is 1. So the labour-cost of labour is 1.

Marx did not reason in terms of input-output models in states of self-replacing equilibrium. Overall, his analysis is irreducibly dynamic and sequential, rather than self-replacing and simultaneous. In the special case of equilibrium it just so happens that the composition and size of the real-wage is irrelevant to the determination of the index-cost of labour. In a circular flow a 'definite quantity of the average labour of society' is precisely equal to that very same quantity of average labour. There is no escape from circularity. For example, in *every* state of self-replacing equilibrium the production of 1 hour of labour-time uses-up 1 hour of labour-time (full capacity utilisation under the vertical integration interpretation); alternatively, in *every* state of self-replacing equilibrium 1 hour of labour-time is exchanged for 1 hour of labour-time (non-arbitrage condition under the market exchange-rate interpretation). To think otherwise is a conceptual error. Compare measuring the length of a metre in terms of metres. Index-cost is fixed in equilibrium and invariant; it therefore is an *absolute not relative* measure. In this sense, any index-cost is an invariable 'measuring rod'.

The convention of unit index-costs ultimately derives from the theoretical abstraction that all instances of a commodity-type are qualitatively identical. To assume that a unit of corn is identical to another unit of corn is relatively harmless; to assume that an hour of labour is qualitatively identical to another hour requires more thought and justification (compare Krause's critique of the 'dogma of homogeneous labour' [35] with Rubin's [65] concept of abstract labour, analysed in a simplified dynamic context in [82]). But as homogeneous labour is a common assumption of linear production theory we will be content to simply assume it here. The convention of unit index-cost also derives from the theoretical abstraction that all instances of a commodity-type are produced under quantitatively identical circumstances. This holds in unchanging equilibrium states, in which the distinction between replacement cost and historical cost is effaced, but not between equilibrium states, in which replacement cost and historical cost diverge. But this leads to dynamics, which we do not consider. The concept of unit index-cost differs in content but not in form from Smith's famous proposition that 'equal quantities of labour, at all times and places, may be said to be of equal value to the labourer' [74]. The main difference is that index-cost is objective; it has no relation to mental state.

In contrast, the Sraffian real-cost 'measuring rod' is not properly constructed. The Sraffian value of labour-power, $\mathbf{v}\bar{\mathbf{w}}^{\mathrm{T}}$, is *not* invariant to a change in income distribution or technology. In general, index-costs are not 1 in Sraffian accounting. A change in the configuration of the economy *changes* the size of the value 'measuring rod'. Compare measuring the length of a road while the length of a metre varies. Sraffa's standard commodity is designed to be a 'measuring rod' that is invariant to changes in the distribution of income (but not technical change). But once conservative real-cost accounting is adopted there is no need for the standard commodity. Comparative real-cost evaluation of commodities in two arbitrary Sraffian systems Ψ_1 and Ψ_2 is potentially both simpler and more general compared to the Sraffian approach. But this is a topic for future work. The implications of Marx's view that Ricardo's 'problem of an "invariable measure of value" was simply a spurious name for the quest for the concept, the nature, of value itself' [46] deserves a separate analysis.

10. Conclusion

The two main conclusions of this paper are: (i) the value-theoretic principle of real-cost is implicit in Sraffa's theoretical framework (prices of production are proportional to real-costs, corollary 6.3), and (ii) when this principle is applied to Marx's labour theory of value the transformation problem is solved (Marx's conservation theorem 8.4).

The 'value controversy' is the history of a real-cost accounting error that omits the labour-cost of money-capital. In general, Marxists assume price-value divergence but maintain aggregrate conservation of value in price, whereas neo-Ricardians assume price-value divergence but deny aggregrate conservation of value in price. Yet under conditions of self-replacing equilibrium both sides of the argument are mistaken: there is no price-value divergence and there is aggregate conservation. Overall the value controversy is resolved in favour of Marx's theory of value.

To rediscover the value-theoretic principle of real-cost required a root-and-branch conceptual critique of two major assumptions of Sraffa's system: (i) the choice of an undistributed surplus at the cost of an interrupted circular flow, and (ii) the omission of money-capital as a commodity that enters into the production of other commodities. By closing Sraffa's surplus equations we conserve Marx. Sraffa's theoretical contribution is therefore key to the further development of the classical theory of value, which found its highest expression, not in Ricardo, but Marx. For it was Marx, in opposition to Ricardo, who maintained that the contradiction between the law of value and uniform profits was only apparent. Although Marx did not complete his transformation procedure his scientific instincts on this point were nonetheless sound.

Appendices

Appendix A. Proof of lemma 4.8

Proof. First, recall that the total money-capital supplied is $M = \mathbf{q}\bar{\mathbf{m}}^{\mathrm{T}}$. Then

$$M = \mathbf{p}\mathbf{A}^{+}\mathbf{q}^{\mathrm{T}},$$

by equation (11). Expanding,

$$M = \mathbf{p}\mathbf{A}\mathbf{q}^{\mathrm{T}} + \mathbf{p}\bar{\mathbf{w}}^{\mathrm{T}}\mathbf{l}\mathbf{q}^{\mathrm{T}}.$$

Then

$$\begin{aligned} M &= \mathbf{p}\mathbf{A}\mathbf{q}^{\mathrm{T}} + \frac{1}{\mathbf{q}\mathbf{l}^{\mathrm{T}}}\mathbf{p}\mathbf{w}^{\mathrm{T}}\mathbf{l}\mathbf{q}^{\mathrm{T}} && (56) \\ &= \mathbf{p}\mathbf{A}\mathbf{q}^{\mathrm{T}} + \mathbf{p}\mathbf{w}^{\mathrm{T}}, && (57) \end{aligned}$$

by lemma 4.3.

Second, multiply both sides of the Sraffian quantity equation (6) by prices $\mathbf{p}$,

$$\mathbf{p}\mathbf{c}^{\mathrm{T}} = \mathbf{p}\mathbf{q}^{\mathrm{T}} - \mathbf{p}\mathbf{A}\mathbf{q}^{\mathrm{T}} - \mathbf{p}\mathbf{w}^{\mathrm{T}}, \tag{58}$$

and substitute (57) into (58) to get

$$\mathbf{p}\mathbf{c}^{\mathrm{T}} = \mathbf{p}\mathbf{q}^{\mathrm{T}} - \bar{\mathbf{m}}\mathbf{q}^{\mathrm{T}}.$$

Then

$$\begin{aligned} \mathbf{p}\mathbf{c}^{\mathrm{T}} &= \bar{\mathbf{m}}\mathbf{q}^{\mathrm{T}}(1+r) - \bar{\mathbf{m}}\mathbf{q}^{\mathrm{T}} \\ &= \bar{\mathbf{m}}\mathbf{q}^{\mathrm{T}}r, \end{aligned}$$

by equation (10). But $M = \mathbf{q}\bar{\mathbf{m}}^{\mathrm{T}}$; therefore

$$\mathbf{p}\mathbf{c}^{\mathrm{T}} = Mr.$$

By lemma 4.7,

$$\mathbf{p}\bar{\mathbf{c}}^{\mathrm{T}} = r.$$

□

Appendix B. Proof of proposition 5.1

Proof. Let n denote the nth index in sequence S. Consider the edge from node $\{n, n+1\}$ in the real-cost graph. The real-cost associated with the edge, or edge-cost, is $k_{n+1,n}$. By lemma 5.3, $k_{n+1,n} = k_{b,n}k_{n+1,b}$ where b is the start and end node. So edge-cost $\{n, n+1\}$ is equivalent to the edge-cost $\{n, b\}$ multiplied by edge-cost $\{b, n+1\}$.

Consider edge $\{n+1, n+2\}$ with edge-cost $k_{n+2,n+1} = k_{b,n+1}k_{n+2,b}$. By lemma 5.5, the edge-cost $\{n, n+2\}$ is $k_{n+2,n} = k_{n+1,n}k_{n+2,n+1} = k_{b,n}k_{n+2,b}$. Hence edge-cost $\{n, n+1\}$ multiplied by edge-cost $\{n+1, n+2\}$ is equal to edge-cost $\{n, b\}$ multiplied by edge-cost $\{b, n+2\}$.

By induction, multiplying the edge-costs of sequence $\{n, n+1, \ldots, n+m\}$ is equivalent to multiplying the edge-costs of sequence $\{n, b, n+m\}$. For any closed loop S of length m, $n = b$ and $n + m = b$. Thus multiplying the edge-costs reduces to the multiplying the edge-costs of sequence $\{b, b, b\}$, which is $k_{b,b} k_{b,b} = 1$. □

Appendix C. Proof of theorem 6.2

Proof. By equation (23), circular flow quantities, when expressed in terms of quantity of supplied capital, are

$$\mathbf{q}^{\circ}[n+2] = \mathbf{q}^{\circ}[n+2] \begin{bmatrix} [\mathbf{A}^{\circ}]^{\mathrm{T}} & [\mathbf{l}^{\circ}]^{\mathrm{T}} \\ \bar{\mathbf{w}}^{\circ} & 0 \end{bmatrix} + \begin{bmatrix} \bar{\mathbf{c}}^{\circ} & 0 \end{bmatrix} M^{\circ}, \tag{59}$$

where $M^{\circ} = q^{\circ}_{n+2}$ by definition. Let $\mathbf{q}^{\mathrm{c}} = \mathbf{q}^{\circ}[n+1, n+2]$ be the quantities of all commodities, excluding the amount of labour and money capital. Then (59) can be written as

$$\begin{bmatrix} \mathbf{q}^{\mathrm{c}} & L^{\circ} \end{bmatrix} = \begin{bmatrix} \mathbf{q}^{\mathrm{c}}[\mathbf{A}^{\circ}]^{\mathrm{T}} + \bar{\mathbf{w}}^{\circ} L^{\circ} & \mathbf{q}^{\mathrm{c}}[\mathbf{l}^{\circ}]^{\mathrm{T}} \end{bmatrix} + \begin{bmatrix} \bar{\mathbf{c}}^{\circ} M^{\circ} & 0 \end{bmatrix},$$

where $L^{\circ} = q^{\circ}_{n+1}$ by definition. Hence, we get two equations

$$\mathbf{q}^{\mathrm{c}} = \mathbf{q}^{\mathrm{c}}[\mathbf{A}^{\circ}]^{\mathrm{T}} + \bar{\mathbf{w}}^{\circ} L^{\circ} + \bar{\mathbf{c}}^{\circ} M^{\circ} \tag{60}$$

$$L^{\circ} = \mathbf{q}^{\mathrm{c}}[\mathbf{l}^{\circ}]^{\mathrm{T}}. \tag{61}$$

Similarly, by equation (23), circular flow quantities, when expressed in terms of the quantity of labour, are

$$\begin{aligned} \mathbf{q}^{\circ}[n+1] &= \mathbf{q}^{\circ}[n+1] \begin{bmatrix} [\mathbf{A}^{\circ}]^{\mathrm{T}} & [\bar{\mathbf{m}}^{\circ}]^{\mathrm{T}} \\ \bar{\mathbf{c}}^{\circ} & 0 \end{bmatrix} + \begin{bmatrix} \bar{\mathbf{w}}^{\circ} & 0 \end{bmatrix} L^{\circ} \\ \begin{bmatrix} \mathbf{q}^{\mathrm{c}} & M^{\circ} \end{bmatrix} &= \begin{bmatrix} \mathbf{q}^{\mathrm{c}}[\mathbf{A}^{\circ}]^{\mathrm{T}} + \bar{\mathbf{c}}^{\circ} M^{\circ} & \mathbf{q}^{\mathrm{c}}[\bar{\mathbf{m}}^{\circ}]^{\mathrm{T}} \end{bmatrix} + \begin{bmatrix} \bar{\mathbf{w}}^{\circ} L^{\circ} & 0 \end{bmatrix} \end{aligned}$$

Hence

$$M^{\circ} = \mathbf{q}^{\mathrm{c}}[\bar{\mathbf{m}}^{\circ}]^{\mathrm{T}}. \tag{62}$$

By definition $\mathbf{w}^{\circ} = \bar{\mathbf{w}}^{\circ} L^{\circ}$ and $\mathbf{c}^{\circ} = \bar{\mathbf{c}}^{\circ} M^{\circ}$. Hence, equation (60) can be written as

$$\mathbf{q}^{\mathrm{c}} = \mathbf{q}^{\mathrm{c}}[\mathbf{A}^{\circ}]^{\mathrm{T}} + \mathbf{w}^{\circ} + \mathbf{c}^{\circ}. \tag{63}$$

Consider the Sraffian system Ψ. By theorem 5.1 Ψ determines a circular flow $\mathbf{C}_{\Psi}$. Hence, equation (63) corresponding to $\mathbf{C}_{\Psi}$ is identically the quantity equation of the Sraffian system Ψ and $\mathbf{q} = \mathbf{q}^{\mathrm{c}}$.

Also, $L = \mathbf{q}\mathbf{l}^{\mathrm{T}} = \mathbf{q}^{\mathrm{c}}[\mathbf{l}^{\circ}]^{\mathrm{T}} = L^{\circ}$ and $M = \mathbf{q}\bar{\mathbf{m}}^{\mathrm{T}} = \mathbf{q}^{\mathrm{c}}[\bar{\mathbf{m}}^{\circ}]^{\mathrm{T}} = M^{\circ}$.

Combining, $[\mathbf{q}, L, M] = \mathbf{q}^{\circ}$, as required. □

Appendix D. Corn economy numerical example

In this section Sraffian labour-cost accounting is compared to real-cost accounting in the context of a simple corn economy, which has the pedagogical advantage that the dimensionality of the numerical example is at a minimum. Note however that the results hold for arbitrary numbers of commodities.

First, we examine the case of simple commodity production, in which Sraffian labour-costs are consistent with real-costs. Second, we examine the case of capitalist simple reproduction, in which the Sraffian real-cost accounting error is apparent.

D.1. **Simple commodity production.** Consider a corn economy in which 1 ton of corn and 2000 hours of labour are required to produce 5 tons of corn:

Corn in	Labour in	Corn out
1	2000	5

Capitalists are absent so workers consume all the net product.

The technique is $\mathbf{A} = a_{1,1} = 1/5$, a degenerate 1×1 matrix, and direct labour coefficients $\mathbf{l} = l_1 = 400$.

D.1.1. *Sraffian labour accounting.* Sraffian labour-cost accounting is

$$\begin{aligned} \mathbf{v} &= \mathbf{vA} + \mathbf{l} \\ v_1 &= v_1 \frac{1}{5} + 400, \end{aligned}$$

which solves to give a labour-cost of $v_1 = 500$ for corn.

D.1.2. *Real-cost labour accounting.* Instead of directly applying equation (43) of lemma 8.1 let's start from the Sraffian open representation, construct the circular flow representation, and then apply the principle of real-cost. In this case of simple commodity production, Sraffian labour accounting is consistent with real-cost labour accounting (corollary 8.2), so we expect that this procedure will yield the same result of $\tilde{v}_1 = 500$ for corn.

Definition 4.3 The Sraffian system is given by

$$\begin{aligned} \Psi &= (\mathbf{A}, \mathbf{l}, \mathbf{w}, \mathbf{c}) \\ &= (a_{1,1}, l_1, w_1, c_1) \\ &= (1/5, 400, 4, 0), \end{aligned}$$

as the real-wage is 4 tons of corn (1 ton to replace the stock of corn), and there is no capitalist consumption.

Lemma 4.1 Composition of gross output is

$$\begin{aligned} \mathbf{q} &= \mathbf{qA}^{\mathrm{T}} + \mathbf{w} + \mathbf{c} \\ q_1 &= q_1 a_{1,1} + w_1 \\ &= q_1 \frac{1}{5} + 4, \end{aligned}$$

which solves to give $q_1 = 5$, as expected.

Lemma 4.2 Total quantity of labour is

$$\begin{aligned} L &= \mathbf{ql}^{\mathrm{T}} \\ &= q_1 l_1 \\ &= 2000. \end{aligned}$$

Lemma 4.3 Worker consumption coefficients (or real-wage per hour) are

$$\begin{aligned}\bar{\mathbf{w}} &= \mathbf{w}/L \\ \bar{w}_1 &= w_1/L \\ &= 1/500.\end{aligned}$$

Theorem 5.1 The circular flow representation is

$$\mathbf{C}_\Psi = \begin{bmatrix} \mathbf{A} & \bar{\mathbf{w}}^{\mathrm{T}} & \bar{\mathbf{c}}^{\mathrm{T}} \\ \mathbf{l} & 0 & 0 \\ \bar{\mathbf{m}} & 0 & 0 \end{bmatrix} = \begin{bmatrix} a_{1,1} & \bar{w}_1 \\ l_1 & 0 \end{bmatrix} = \begin{bmatrix} 1/5 & 1/500 \\ 400 & 0 \end{bmatrix}.$$

Lemma 5.4 Real-costs are given by

$$\mathbf{k}_j[j] = \mathbf{C}_{(j)}[j]\mathbf{L}_j.$$

Hence labour cost of corn is

$$\begin{aligned}\tilde{v}_1 = k_{1,2} &= \mathbf{C}_{(2)}[2]\mathbf{L}_2 \\ &= l_1(1 - a_{1,1})^{-1} \\ &= 500,\end{aligned}$$

as expected.

Lemma 5.5 The corn-cost of labour is 1/500.

D.1.3. *Summary.* In simple commodity production, Sraffian labour accounting is consistent with real-cost accounting. Both give $v_1 = \tilde{v}_1 = 500$ hours.

D.2. **Simple reproduction.** Consider the economy

Corn in	Labour in	Corn out
2	4000	10

.

Assume that workers consume the same real-wage as before, $w_1 = 4$, and a capitalist class consumes the remainder of the net product.

As before, the technique is $\mathbf{A} = a_{1,1} = 1/5$, a degenerate 1×1 matrix, and the direct labour coefficients are $\mathbf{l} = l_1 = 400$.

D.2.1. *Sraffian labour accounting.* Sraffian labour accounting in simple reproduction is identical to the case of simple commodity production,

$$\begin{aligned}\mathbf{v} &= \mathbf{vA} + \mathbf{l} \\ v_1 &= v_1 \frac{1}{5} + 400,\end{aligned}$$

which gives the same answer, $v_1 = 500$ for corn, as before.

D.2.2. *Real-cost labour accounting.* In the case of simple reproduction, Sraffian labour accounting is not consistent with real-cost labour accounting (Lemma 8.1), as we will now demonstrate.

Again, we'll take a longer route and avoid directly applying equation (54) of Lemma 8.5. Instead we'll start from the Sraffian open representation, construct the circular flow representation, and then apply the principle of real-cost.

Definition 4.3 The Sraffian system is

$$\begin{aligned} \Psi &= (\mathbf{A}, \mathbf{l}, \mathbf{w}, \mathbf{c}) \\ &= (a_{1,1}, l_1, w_1, c_1) \\ &= (1/5, 400, 4, 4), \end{aligned}$$

as capitalists consume 4 tons of corn.

Lemma 4.1 Composition of gross output is

$$\begin{aligned} \mathbf{q} &= \mathbf{q}\mathbf{A}^{\mathrm{T}} + \mathbf{w} + \mathbf{c} \\ q_1 &= q_1 a_{1,1} + w_1 + c_1 \\ &= q_1 \frac{1}{5} + 4 + 4, \end{aligned}$$

which solves to give $q_1 = 10$, as expected.

Lemma 4.2 Total quantity of labour is

$$\begin{aligned} L &= \mathbf{q}\mathbf{l}^{\mathrm{T}} \\ &= q_1 l_1 \\ &= 4000. \end{aligned}$$

Lemma 4.3 Worker consumption coefficients (or real-wage per hour) are

$$\begin{aligned} \bar{\mathbf{w}} &= \mathbf{w}/L \\ \bar{w}_1 &= w_1/L \\ &= 1/1000. \end{aligned}$$

Note that the real-wage per hour is smaller in this economy compared to the previous economy.

Lemma 4.4 The rate of profit is $r = (1/\lambda_*) - 1$, where

$$\begin{aligned} \mathbf{p}\mathbf{A}^{+} &= \lambda_* \mathbf{p} \\ \mathbf{p}(\mathbf{A} + \bar{\mathbf{w}}^{\mathrm{T}}\mathbf{l}) &= \lambda_* \mathbf{p} \\ p_1(a_{1,1} + \bar{w}_1 l_1) &= \lambda_* p_1, \end{aligned}$$

which gives $\lambda_* = 3/5$. Hence $r = 2/3$ (= 66.7% as an advance of 6 units of corn yields a profit of 4 units of corn).

Lemma 4.5 The money-capital coefficients are determined up to a *numéraire* equation

$$\begin{aligned} \bar{\mathbf{m}} &= \lambda_* \mathbf{p} \\ \bar{m}_1 &= \frac{3p_1}{5}. \end{aligned}$$

This dependency is later eliminated as real-cost labour values are independent of the choice of *numéraire.*

Lemma 4.6 The total money-capital is

$$\begin{aligned} M &= \mathbf{q}\bar{\mathbf{m}}^{\mathrm{T}} \\ &= q_1 \bar{m}_1 \\ &= 6p_1. \end{aligned}$$

Lemma 4.7 Capitalist consumption coefficients are

$$\begin{aligned} \bar{\mathbf{c}} &= \mathbf{c}/M \\ \bar{c}_1 &= c_1/M \\ &= \frac{2}{3p_1}. \end{aligned}$$

Theorem 5.1 The circular flow representation is

$$\mathbf{C}_\Psi = \begin{bmatrix} \mathbf{A} & \bar{\mathbf{w}}^{\mathrm{T}} & \bar{\mathbf{c}}^{\mathrm{T}} \\ \mathbf{l} & 0 & 0 \\ \bar{\mathbf{m}} & 0 & 0 \end{bmatrix} = \begin{bmatrix} a_{1,1} & \bar{w}_1 & \bar{c}_1 \\ l_1 & 0 & 0 \\ \bar{m}_1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1/5 & 1/1000 & \frac{2}{3p_1} \\ 400 & 0 & 0 \\ \frac{3p_1}{5} & 0 & 0 \end{bmatrix}.$$

Lemma 5.4 Real-costs are given by

$$\mathbf{k}_j[j] = \mathbf{C}_{(j)}[j]\mathbf{L}_j.$$

Hence the labour-cost of corn and the labour-cost of money-capital is

$$\begin{aligned} [\ \tilde{v}_1 \quad \omega\ ] = \mathbf{k}_2[2] &= \mathbf{C}_{(2)}[2]\mathbf{L}_2 \\ &= \mathbf{C}_{(2)}[2](\mathbf{I} - \mathbf{C}[2|2])^{-1} \\ &= [\ l_1 \quad 0\ ] \begin{bmatrix} 4/5 & -\frac{2}{3p_1} \\ -\frac{3p_1}{5} & 1 \end{bmatrix}^{-1} \\ &= [\ l_1 \quad 0\ ] \frac{5}{2} \begin{bmatrix} 1 & \frac{2}{3p_1} \\ \frac{3p_1}{5} & \frac{4}{5} \end{bmatrix} \end{aligned}$$

Hence,

$$\tilde{v}_1 = \frac{5}{2} l_1 = 1000.$$

So the labour-cost of corn is $\tilde{v}_1 = 1000$, which differs from the Sraffian result of $v_1 = 500$, as expected.

Lemma 5.5 The corn-cost of labour is 1/1000. (Compared to simple commodity production the workers work more hours per unit of corn consumed).

D.2.3. *Summary.* Sraffian labour-values (v_1) and real-cost labour-values ($\tilde{v}_1$) are consistent in simple commodity production because the commodity money-capital is absent.

Sraffian labour-values and real-cost labour-values are inconsistent in capitalist simple reproduction because Sraffian accounting omits the the labour-cost of money-capital. The labour-cost of capitalist consumption is not accounted for.

Perhaps the best way to see this is to consider the reduction to 'dated' labour representation, in which each term of the expansion of the labour-cost equation is considered to represent production in a previous 'round'.

Sraffa's 'dated' labour representation is

$$\mathbf{v} = \mathbf{l}(\mathbf{I} - \mathbf{A})^{-1} \tag{64}$$

$$v_1 = l_1 \left[1 + a_{1,1} + a_{1,1}^2 + a_{1,1}^3 + \ldots\right]. \tag{65}$$

The usual interpretation of this expansion is that the labour-value of corn is the sum of all the labour required in all the previous 'rounds'. Reading the terms in the expansion we can see that to produce 1 unit of corn requires l_1 labour in the last round, plus $l_1 a_{1,1}$ labour in the previous 'round' to produce the corn input, and so on, 'vertically integrating' back up the causal chain until the series converges. But Sraffian accounting does not count all the labour, as we shall now see.

Lemma 8.1 In simple reproduction real-cost values may be written as,

$$\tilde{\mathbf{v}} = \mathbf{l}(\mathbf{I} - \tilde{\mathbf{A}})^{-1},$$

where $\tilde{\mathbf{A}} = \mathbf{A} + \bar{\mathbf{c}}^{\mathrm{T}}\bar{\mathbf{m}}$ is the technique augmented by capitalist consumption.

$$\begin{aligned}
\tilde{\mathbf{A}} &= \mathbf{A} + \bar{\mathbf{c}}^{\mathrm{T}}\bar{\mathbf{m}} \\
\tilde{a}_{1,1} &= a_{1,1} + \bar{c}_1 \bar{m}_1 \\
\tilde{a}_{1,1} &= a_{1,1} + \frac{2}{3p_1}\frac{3p_1}{5} \\
&= a_{1,1} + \frac{2}{5}.
\end{aligned}$$

The real-cost 'dated' labour representation is therefore

$$\tilde{\mathbf{v}} = \mathbf{l}(\mathbf{I} - \tilde{\mathbf{A}})^{-1} \tag{66}$$

$$\tilde{v}_1 = l_1 \left[1 + (a_{1,1} + \frac{2}{5}) + (a_{1,1} + \frac{2}{5})^2 + (a_{1,1} + \frac{2}{5})^3 + \ldots\right]. \tag{67}$$

Compare equation (67) to equation (65). The extra $2/5$ term, absent in Sraffian accounting, augments $a_{1,1}$ and represents the corn required to supply the money-capital necessary to produce 1 unit of corn. In other words, capitalists consume 4 units of corn per 10 units of corn produced. This corn is a socially necessary cost of production under capitalist conditions.

Real-cost labour-values count the labour-time required to produce the capitalist consumption bundle whereas Sraffian labour-values do not. This labour-time is missing in Sraffian accounting because money-capital is not considered a commodity with an associated real-cost. Sraffian labour-cost accounting is consistent with the principle of real-cost only in simple commodity production, that is production absent a capitalist class.

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