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Convergence to natural prices in simple production

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This paper describes a nonlinear dynamic model of the convergence of market prices to natural prices in a multisector 'simple production' economy under conditions of a constant technique and composition of demand. Prices and quantities adjust in real time according to the classical principle of cross-dual dynamics. The economy gravitates toward an asymptotically stable equilibrium in which natural prices are proportional to labor-values. To demonstrate an application of the model we reply to Mirowski's (1989) critique that Marx held a contradictory 'substance' and 'field' theory of value.

I. THE CLASSICAL PROCESS OF GRAVITATION TOWARD NATURAL PRICES

The coordination of millions of independent production activities in a large-scale market economy is neither perfect or equitable but nonetheless 'one should be far more surprised by the existing degree of coordination than by the elements of disorder' (Boggio, 1995). The classical Political Economists developed a theoretical framework in which this surprising fact could be understood.

Reproducible commodities are those 'that may be multiplied ... almost without any assignable limit, if we are disposed to bestow the labor necessary to obtain them' (Ricardo, 1996). Classical authors, such as Smith, Ricardo, and Marx, argued that the market prices of reproducible commodities tend to gravitate toward or around their natural prices (e.g., Smith (1994), Book 1, Chapter VII) or 'prices of production' (Marx, 1971). Natural prices are relatively stable prices robust to 'accidental and temporary deviations' (Ricardo, 1996) between supply and demand that manifest when quantities supplied equal quantities demanded. On this view, market prices are short-term, out-of-equilibrium prices that arise from imbalances between supply and demand whereas natural prices are long-term, equilibrium prices that derive from the objective conditions of production. For example, Ricardo (1996) writes, 'It is the cost of production which must ultimately regulate the price of commodities, and not, as has often been said, the proportion between supply and demand: the proportion between supply and demand may, indeed, for a time, affect the market value of a commodity, until it is supplied in greater or less abundance, according as the demand may have increased or diminished; but this effect will only be of temporary duration'.

The process of gravitation toward natural prices is an unintended consequence of the self-interested decisions of economic actors engaged in competition. Capital gets withdrawn from unprofitable sectors and reallocated to profitable sectors. Supply increases with additional capital whereas it decreases with the withdrawal of capital. Increased supply acts negatively on prices and profitability, whereas decreased supply acts positively. Given the theoretical assumption that the determinants of natural prices remain constant, such as the productivity of labor, then the process continues until a general or average rate of profit prevails and the incentive to reallocate capital has gone. At this point, the forces of supply and demand are in balance, and market prices equal natural prices. For example, Marx (1971, pg. 366) writes, 'the general rate of profit is never anything more than a tendency, a movement to equalize specific rates of profit. The competition between capitalists – which is itself this movement toward equilibrium – consists here of their gradually withdrawing capital from spheres in which profit is for an appreciable length of time below average, and gradually investing capital into spheres in which profit is above average'.

Although there are important differences between the classical economists this view of the homeostatic kernel of capitalist competition is more-or-less shared by them. For example, Smith, Ricardo and Marx all claim that, in the absence of profit on capital and rent on land, natural prices are proportional to labor-values, which measure the objective costs of production in terms of labor-time.

The classical economists did not develop formal dynamic models of the process of gravitation. Marx, however, embarked on a close and extensive study of the calculus (Marx, 1983), since he believed that mathematics held the promise of 'determining the main laws of capitalist crisis' (Marx, Letter to Engels, May 31, 1873, quoted by Kol'man and Yanovskaya (1983)). But Marx's quantitative models remained small-scale numerical examples of simultaneous equations (e.g., his Volume 2 reproduction schemes (Marx, 1974); see Trigg (2006) for a modern elaboration) or numerical examples of two-step iteration (e.g., his Volume 3 discussion of the transformation of values to prices of production; see Shaikh (1977) for a modern elaboration). He did not develop economic models that featured – for example – differential equations.

II. MODERN DYNAMIC MODELS OF GRAVITATION

Morishima (e.g., 1990, pg. 84) dubbed the classical adjustment process 'cross-dual'; 'dual' because adjustment includes simultaneous changes in both prices and quantities, and 'cross' because imbalances between quantities supplied and demanded entail price changes, and imbal-

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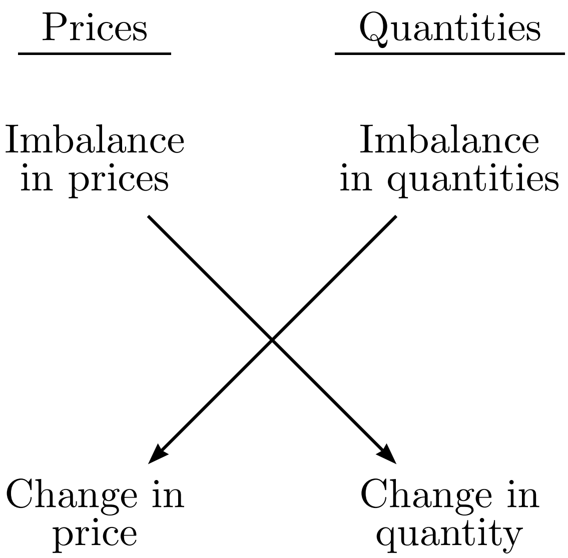


FIG. 1. Classical cross-dual adjustment

ances between costs and revenues entail quantity changes (see Figure 1). The term 'cross-dual' also serves to demarcate the classical process of gravitation from neoclassical tâtonnement that occurs 'out of time' with pure price adjustment and infinitely fast, or instantaneous, quantity adjustment (e.g., Varian (1992, pg. 398) and Tuinstra (2001), and also see Flaschel *et al.* (1997, ch. 2) and Flaschel (2010) for a discussion of the differences). However, it is worth pointing out that Walras also formulates a cross-dual dynamics in the context of a production economy, recently formalized by Mas-Collel (1986).

Since the 1950s a small number of economists have applied dynamical systems theory in order to develop the classical theory of gravitation. The main obstacle to this kind of work is the analytical intractability of large-scale systems of nonlinear differential equations. Although a rich mathematical theory of dynamics exists there are no fully general methods for solving and analyzing such systems.

A key question asked by modern studies is whether formal dynamical models of cross-dual adjustment converge toward a stable equilibrium (see Steedman (1984) for an early survey and also more recently Flaschel (2010)). The results are mixed, ranging from instability (i.e., lack of convergence) to stability (i.e., bounded orbits around natural prices) to asymptotic stability (i.e., convergence to natural prices). The mixed stability results reflect the variety of models developed under the rubric of cross-dual dynamics (e.g., see the collection edited by Semmler (1985)). For example, Flaschel (2010, ch. 15) proves that a cross-dual model of an economy on a balanced growth path with constant returns to scale, joint-production and a constant composition of demand is stable. If the adjustment rules are modified so that capital reallocation takes into account the rate of change of profit (rather than simply the size of profit) then the model is globally asymptotically stable.

This working paper is a preliminary technical exercise in the elaboration and analysis of a multisector cross-dual model restricted to the case of 'simple production', a theoretical simplification in which workers receive wages and share out-of-equilibrium residual profits but the the ownership of capital and land is absent. This paper deepens and generalizes an earlier analysis of the law of value (Wright, 2008) in the context of 'simple commodity production' (Rubin, 1973).

My overall aim is to work toward a formal, dynamic framework that can be used to study, in forensic detail, the conceptual controversies associated with the classical labor theory of value, rather than restricting analysis to the standard (and insufficient) framwork of static, linear production models (e.g., see Sraffa (1960); Samuelson (1971); Steedman (1981); Keen (1998); Cockshott *et al.* (2009); Wright (2009)). Cross-dual dynamic models include standard linear production models as special cases but more importantly are inherently more faithful representations of classical thought, especially Marx's irreducibly dynamic analysis. This paper is a first step in this direction.

The paper is organized as follows: we begin by describing the dynamic equations, and then eliminate variables to produce a reduced form of the model; then we characterize the equilibrium and prove that it is asymptotically stable; then we present a numerical example that gives an indication of the kinds of economic trajectories that are possible; and finally we apply the model to reply to Mirowski's critique of Marx's labor theory of value.

III. A MODEL OF SIMPLE PRODUCTION

Assume $n \in \mathbb{Z}^+$ sectors that specialize in the production of one commodity type or category. The technique is a non-negative $n \times n$ input-output matrix of inter-sector coefficients, $\mathbf{A} = [a_{i,j}]$. Each $a_{i,j} \geq 0$ is the quantity of commodity i directly required to output 1 unit of commodity j. Matrix $\mathbf{A}$ is fully connected and $\mathbf{I} - \mathbf{A}$ is of full rank. There is a vector $\mathbf{x}^{\mathrm{T}} \in \mathbb{R}^n_+$ such that $\mathbf{x}^{\mathrm{T}} > \mathbf{A}\mathbf{x}^{\mathrm{T}}$; that is, the technique is productive. All vectors are row vectors. The direct labor coefficients are a $1 \times n$ vector, $\mathbf{l} = [l_i]$. Each $l_i > 0$ is the quantity of labor directly required to output 1 unit of commodity i.

Assume constant returns to scale so $\mathbf{A}$ and $\mathbf{l}$ are fixed throughout. There are two motivations for this abstraction. First, we can study market adjustment under the theoretically ideal condition of an unchanging objective cost structure. Second, price and quantity adjustments are fast compared to technical change. So an analysis of the fast dynamics while assuming fixed slow dynamics (i.e., decoupling price and quantity adjustment from technical change) may transfer to the more general case of ongoing technical change, non-constant returns to scale and growth (c.f. the method of adiabatic approximation in the physical sciences). In other words, we avoid trying to change direction before we have mastered the ability to walk in a straight line.

The constant L denotes the size of the labor force, which is the only fixed resource. All variables are a function of time unless explicitly declared constant.

III.1. Workers' propensity to consume

The aggregate savings of worker households is a sum of non-commodity money m deposited in the accounts of a notional bank. The workers' propensity to consume is defined as a constant fraction, $\alpha \in (0, 1]$, of this sum. The aggregate consumption expenditure demand in the economy is therefore αm and the amount saved $(1-\alpha)m$.

III.2. The real wage

Assume that worker households are flexible with regard to the scale of their consumption but not the commodity bundle they demand. The $1 \times n$ real wage vector, $\mathbf{w}' = [w'_i]$, has a constant composition, denoted by the $1 \times n$ wage composition vector $\mathbf{w} = [w_i]$, but has variable scale; that is $\mathbf{w}' = k\mathbf{w}$ always for some scale factor k. By implication the real wage is always sufficient to ensure the reproduction of the labor force, L.

Market prices are a $1 \times n$ vector $\mathbf{p} = [p_i]$. The real wage is a function of the aggregate demand, αm, and the price of workers' consumption goods, $\mathbf{p}\mathbf{w}^{\mathrm{T}}$. The fraction $\alpha m/\mathbf{p}\mathbf{w}^{\mathrm{T}}$ denotes the number of real wage bundles of composition $\mathbf{w}$ that are purchased at prices $\mathbf{p}$. The real wage is therefore

$$\mathbf{w}' = \frac{\alpha m}{\mathbf{p}\mathbf{w}^{\mathrm{T}}}\mathbf{w}^{\mathrm{T}},$$

where $k = \alpha m/\mathbf{p}\mathbf{w}^{\mathrm{T}}$ is the variable scale factor (composition vector $\mathbf{w}$ defines a ray in commodity space that the real wage is constrained to move along). Given a constant aggregate demand lower (resp. higher) prices imply a higher (resp. lower) real wage.

III.3. Sectoral profit

The revenue received by sector i is the amount of product it sells in the market multiplied by the current price. Demand has two components: demand from other sectors and demand from worker households.

Quantities produced (or sectoral activity levels) are a $1 \times n$ vector $\mathbf{q} = [q_i]$. The demand from other sectors, $\mathbf{A}_{(i)}\mathbf{q}^{\mathrm{T}}$, is a function of the technique and activity levels. The demand from worker households is the ith component of the real wage, $(\alpha m/\mathbf{p}\mathbf{w}^{\mathrm{T}})w_i$. The total demand for commodity i is then

$$D_i = \mathbf{A}_{(i)}\mathbf{q}^{\mathrm{T}} + \frac{\alpha m}{\mathbf{p}\mathbf{w}^{\mathrm{T}}}w_i$$

and therefore the total revenue for sector i is $\rho_i = p_i D_i$.

The cost incurred by sector i during the production of q_i is the amount of inputs bought in the market multiplied by their current prices. Cost also has two components: the cost of input commodities produced by other sectors and wage costs. The cost of input commodities, $\mathbf{p}\mathbf{A}^{(i)}q_i$, is a function of the technique, commodity prices and the activity level. The wage cost, $l_i w q_i$, is a function of the direct labor coefficient, the wage rate and the sectoral activity level. Hence the overall cost incurred by sector i is

$$\kappa_i = q_i(\mathbf{p}\mathbf{A}^{(i)} + l_i w).$$

The instantaneous profit obtained (or loss incurred) within sector i is the difference between revenues and costs, that is $\pi_i = \rho_i - \kappa_i$; or, in full,

$$\pi_i = p_i(\mathbf{A}_{(i)}\mathbf{q}^{\mathrm{T}} + \frac{\alpha m}{\mathbf{p}\mathbf{w}^{\mathrm{T}}}w_i) - q_i(\mathbf{p}\mathbf{A}^{(i)} + l_i w). \qquad (1)$$

III.4. Worker savings

$\mathbf{l}\mathbf{q}^{\mathrm{T}}$ is the level of employment. Workers' aggregate savings, m, are augmented by an inflow of wage payments, $\mathbf{l}\mathbf{q}^{\mathrm{T}}w$, where w is the money wage rate, and reduced by an outflow of consumption spending, which is the fraction αm spent on the real wage.

All firms are owned by workers. So workers share profits from firms or bear the losses incurred by them. Savings m are therefore increased (or decreased) according to total sectoral profits (or losses), $\sum \pi_i$. The rate of change of total savings is therefore the sum of deposits and withdrawals,

$$\frac{\mathrm{d}m}{\mathrm{d}t} = \mathbf{l}\mathbf{q}^{\mathrm{T}}w - \alpha m + \sum_{i=1}^{n}\pi_i. \qquad (2)$$

III.5. The wage rate

The wage rate, w, given a fixed working population L, varies with the demand for labor. Assume a standard Phillips-like (1958) labor market such that the change in the wage rate depends both on the level of unemployment and the rate of change of unemployment. So an increase (resp. decrease) in the level of unemployment, $-\mathbf{l}\frac{\mathrm{d}\mathbf{q}^{\mathrm{T}}}{\mathrm{d}t} > 0$ (resp. < 0) causes a relative wage decrease (resp. increase); that is $\frac{1}{w}\frac{\mathrm{d}w}{\mathrm{d}t} \propto \mathbf{l}\frac{\mathrm{d}\mathbf{q}^{\mathrm{T}}}{\mathrm{d}t}$. In addition, as the level of employment rises, and the labor market tightens, the relative wage also rises, until it approaches ∞ at the hypothetical maximum of full employment; that is, $\frac{1}{w}\frac{\mathrm{d}w}{\mathrm{d}t} \propto \frac{1}{L - \mathbf{l}\mathbf{q}^{\mathrm{T}}}$. Combining these two factors we get

$$\frac{\mathrm{d}w}{\mathrm{d}t} = \lambda_w \mathbf{l}\frac{\mathrm{d}\mathbf{q}}{\mathrm{d}t}^{\mathrm{T}}\frac{1}{L - \mathbf{l}\mathbf{q}^{\mathrm{T}}}w, \qquad (3)$$

where $\lambda_w > 0$ is an arbitrary adjustment coefficient.

III.6. Commodity inventories

Each sector stores a stock of unsold inventories, denoted s_i. The supply of commodity i, q_i, will not in general equal the demand for quantity i, D_i. When supply is greater than (resp. less than) demand then inventories increase (resp. decrease). The rate of change of inventories is the difference between supply and demand, $\frac{\mathrm{d}s_i}{\mathrm{d}t} = q_i - D_i$; or, in full,

$$\frac{\mathrm{d}s_i}{\mathrm{d}t} = q_i - \mathbf{A}_{(i)}\mathbf{q}^{\mathrm{T}} - \frac{\alpha m}{\mathbf{p}\mathbf{w}^{\mathrm{T}}}w_i. \qquad (4)$$

Assume that commodities are imperishable so inventories can be stored indefinitely. A more general model would allow inventories to be destroyed according to a per sector decay rate. Then the inventory held by service sectors could be interpreted as short-term excess capacity, for example due to the ability of service providers to store intermediate products and work with greater intensity.

III.7. Cross-dual price and quantity adjustment

A sector consists of a collection of firms that specialize in the production of the same commodity type or category. A sector's overall price and quantity adjustment is the aggregate of the adjustments of the individual firms that comprise it. So this model is meso-level, sandwiched between the micro-level of individual firms and the macro-level of global aggregates.

III.7.1. Price adjustment

An excess or lack of demand for a commodity translates into a change in the size of inventories. For example, underproduction relative to demand means that inventories shrink, whereas overproduction means that inventories grow. Firms tend to raise prices when inventories shrink on the assumption that buyers will outbid each other to obtain the scarce product, whereas firms tend to lower prices when inventories grow on the assumption that other firms will underbid each other in order to sell to scarce buyers. The sector as a whole, therefore, adjusts the relative price of its commodity in proportion to excess demand, that is $\frac{1}{p_i}\frac{\mathrm{d}p_i}{\mathrm{d}t} \propto -\frac{\mathrm{d}s_i}{\mathrm{d}t}$. A quantity imbalance, represented by the change in inventory size, translates into a price adjustment.

In addition, the magnitude of price adjustment approaches positive ∞ as inventory approaches zero and the commodity is completely scarce, that is $\frac{1}{p_i}\frac{\mathrm{d}p_i}{\mathrm{d}t} \propto \frac{1}{s_i}$. Combining these two factors we get the price adjustment equation

$$\frac{\mathrm{d}p_i}{\mathrm{d}t} = -\lambda_i \frac{\mathrm{d}s_i}{\mathrm{d}t}\frac{p_i}{s_i}, \tag{5}$$

where $\lambda_i > 0$ is an arbitrary adjustment coefficient. Sectors with small (resp. large) inventories will tend to adjust prices relatively quickly (resp. slowly). Assume that firms do not reduce prices to dump inventory and realize value but instead maintain an inventory buffer to manage any variance in excess demand.

III.7.2. Quantity adjustment

We abstract from intermediate accounts, such as working capital held by firms, so the only bank deposits are the private deposits of workers. The bank is the sole source of capital and funds production by advancing to firms the money required to cover the cost of their inputs. Firms sell their product and revenue returns to the bank. The net transfer between a sector and the bank is a flow of profit (or loss), π_i, which either increases or decreases total bank deposits. Interest is not charged and there are no credit mechanisms. The sectors funded by the bank represent a portfolio of n investments. The bank aims to maximize its rate of return by deciding to withdraw capital from loss-making sectors and inject capital into profit-making sectors based on rate of profit signals. Duménil and Lévy (1998) argue that this kind of adjustment behavior is equivalent to intertemporal optimization.

The rate of profit is the ratio of profit, π_i, to cost, $(\mathbf{pA}^{(i)}+l_i w)q_i$. But here we introduce an analytical simplification. The component of cost that represents payment for non-labor inputs, $\mathbf{pA}^{(i)}q_i$, is a transfer of funds from sector i to the other n sectors that supply non-labor inputs (some sectors may use their own product as input and we count this case as an explicit payment from the sector to itself). Hence this component of cost resolves into revenue for other sectors and is directly returned to the bank. In consequence, the total non-labor costs advanced, $\mathbf{pAq}^{\mathrm{T}}$, always return in the form of firm revenue, whereas the wages advanced do not necessarily return in the form of firm revenue. So non-labor costs get netted out of the bank's total advance to fund production and the actual cost to the bank of funding production at scale q_i in sector i is the total wage bill, $l_i q_i w$. Assume, therefore, that the rate of profit the bank considers when allocating capital is $\pi_i/(l_i q_i w)$.

The relative change in the scale of production is proportional to the rate of profit (or loss), that is $\frac{1}{q_i}\frac{\mathrm{d}q_i}{\mathrm{d}t} \propto \frac{\pi_i}{l_i q_i w}$. A price imbalance, represented by the rate of profit, translates into a quantity adjustment. In consequence, we get the quantity adjustment equation

$$\frac{\mathrm{d}q_i}{\mathrm{d}t} = \lambda_q \frac{\pi_i}{l_i w}, \tag{6}$$

where $\lambda_q > 0$ is an arbitrary adjustment coefficient. Sectors with a high (resp. low) rate of profit have capital injected (resp. withdrawn) in order to increase (resp. decrease) the scale of their production.

This completes the phenomenological description of the model; next we will examine the equations more closely to eliminate variables and construct a dynamical system in prices and quantities only.

IV. WAGES AND THE SCALE OF PRODUCTION

The wage rate w can be eliminated and replaced by a function of the scale of production, $\mathbf{q}^{\mathrm{T}}$.

Lemma 1. The wage in terms of the scale of production is

$$w(t) = k_w \frac{1}{(L - \mathbf{lq}^{\mathrm{T}})^{\lambda_w}}, \tag{7}$$

where

$$k_w = w(0)(L - \mathbf{lq}(0)^{\mathrm{T}})^{\lambda_w}$$

is a positive constant.

Proof. Let $\theta = \mathbf{lq}^{\mathrm{T}}$ be the employment level. Wage adjustment equation (3) is then separable,

$$\frac{1}{w}\frac{\mathrm{d}w}{\mathrm{d}t} = \lambda_w \frac{1}{L-\theta}\frac{\mathrm{d}\theta}{\mathrm{d}t}.$$

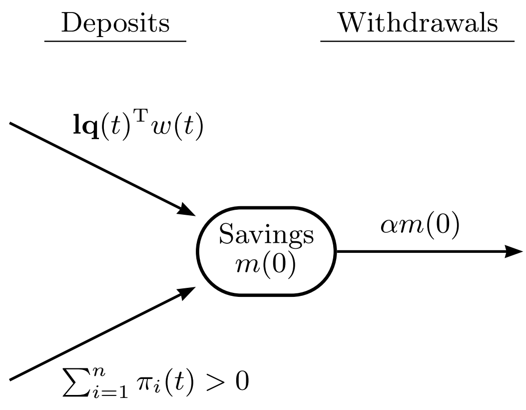


FIG. 2. Aggregate money flows and savings under conditions of aggregate profit.

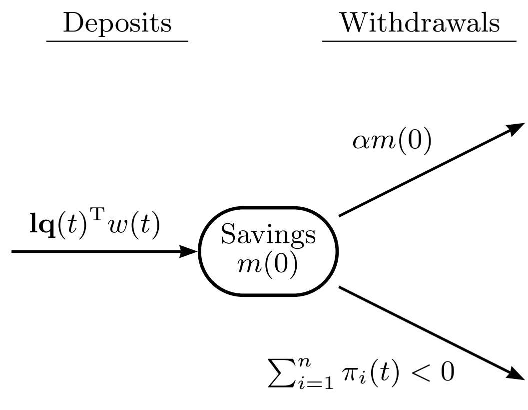


FIG. 3. Aggregate money flows and savings under conditions of aggregate loss.

Integrating

$$\int \frac{1}{w}\mathrm{d}w = \lambda_w \int \frac{1}{L-\theta}\mathrm{d}\theta$$

gives

$$w(t) = k^{\lambda_w}\frac{1}{(L-\mathbf{lq}^{\mathrm{T}})^{\lambda_w}}$$

where k is a constant of integration. At $t = 0$ we have $k^{\lambda_w} = w(0)(L - \mathbf{lq}(0)^{\mathrm{T}})^{\lambda_w}$. □

Equation (7) eliminates the wage rate w from the system of equations. Low (resp. high) unemployment implies high (resp. low) wages.

V. MARKET PRICES AS INDICES OF RELATIVE SCARCITY OR ABUNDANCE

The n inventories variables, s_i, can be eliminated and replaced by functions of the corresponding price, p_i.

Lemma 2. Inventory levels in terms of prices are

$$s_i(t) = p_i(0)^{\frac{1}{\lambda_i}} s_i(0)\frac{1}{p_i^{\lambda_i}} \tag{8}$$

for all i.

Proof. Price adjustment equation (5) is separable. Integrating

$$-\frac{1}{\lambda_i}\int \frac{1}{p_i}\,dp_i = -\int \frac{1}{s_i}\,ds_i$$

gives

$$s_i(t) = k\frac{1}{p_i^{\lambda_i}}$$

where k is a constant of integration. At $t = 0$ we have $k = p_i(0)^{1/\lambda_i}s_i(0)$. □

Equation (8) eliminates inventories s_i from the system of equations. It describes an inverse power law relationship between the level of inventory in sector i and the price of that sector's product, p_i. A high (resp. low) price implies a low (resp. high) inventory level. The monetary value of the variable amount of inventory held by a sector remains constant, $p_i s_i = p_i(0)^{\frac{1}{\lambda_i}} s_i(0)$ for all i. A very natural interpretation of this relationship is that market prices are indices of scarcity (or abundance).

VI. CONSERVATION OF THE MONEY STOCK

The deposits to worker accounts (in the form of wage payments or profits) are exactly equal to withdrawals (in the form of consumption spending or losses). So the total stock of money held by the bank is always constant. We can therefore eliminate the variable $m(t)$ and replace it with the constant $m(0)$.

Lemma 3. Workers savings are constant,

$$m(t) = m(0). \tag{9}$$

Proof. Sum equations (1) to deduce that

$$\sum_{i=1}^{n}\pi_i = \alpha m - \mathbf{lq}^{\mathrm{T}}w. \tag{10}$$

Substitute into equation (2) to get $\frac{\mathrm{d}m}{\mathrm{d}t} = 0$. Hence $m(t) = k$, where k is a constant of integration. At $t = 0$ we have $k = m(0)$. □

Equation (10) expresses an aggregate money conservation identity,

$$\alpha m = \sum_{i=1}^{n}\pi_i + \mathbf{lq}^{\mathrm{T}}w,$$

which states that, for all t, aggregate expenditure is equal to the aggregate profit (or loss) plus the total wage income.

For example, consider the case, depicted in Figure 2, of total aggregate profit in the economy. Workers withdraw the proportion $\alpha m(0)$ to spend on consumption goods. They receive $\mathbf{lq}^{\mathrm{T}} w$ as wage income. The bank funds production and receives a net income of $\sum \pi_i > 0$ that gets added to workers' deposits. Total profit is therefore the excess of aggregate demand over the wage bill.

For example, consider the case, depicted in Figure 3, of total aggregate loss in the economy. As before, workers spend $\alpha m(0)$ on consumption goods and receive $\mathbf{lq}^{\mathrm{T}} w$ as wage income. The bank funds production and makes a net loss of $\sum \pi_i < 0$ that gets withdrawn from workers' deposits. Total losses therefore are the excess of the wage bill over aggregate demand.

The bank is the single hub in the economy where the total stock of money is held. The aggregate conservation of the money stock is a necessary consequence of the local conservation of money during its circulation. Aggregate demand always returns to worker accounts either in the form of wages or profits, $\alpha m(0) = \mathbf{lq}^{\mathrm{T}} + \sum \pi_i$. Money conservation identity (10) plays an important role in proving the convergence of market prices to natural prices.

VII. THE AGGREGATE DEMAND FOR LABOR-TIME

In the context of simple production the replacement costs of commodities, measured in units of labor-time, is the $1 \times n$ vector of labor-values,

$$\mathbf{v} = \mathbf{vA} + \mathbf{l}. \tag{11}$$

Labor-values measure the 'total sum of the labor directly and indirectly expended on the production of any product *under present-day production conditions*' independent of any 'historical digressions' regarding the past state of the economy (Dmitriev (1974), pp. 43–44). Equation (11) is consistent with Marx's description that the total labor-time required to produce a commodity is a sum of 'dead' or indirect labor 'embodied' in means of production ($\mathbf{vA}$) plus an addition of 'living' or direct labor ($\mathbf{l}$) (Marx, 1999, 1954). The indirect labor-value of used-up input commodities is transferred whereas the direct labor is added to the product; for example, Marx (1954, pg. 193) writes, 'the values of the means of production used up in the process [of production] are preserved, and present themselves afresh as constituent parts of the value of the product'. The solution of equation (11) can be represented in terms of a Leontief inverse, $\mathbf{v} = \mathbf{l}(\mathbf{I} - \mathbf{A})^{-1}$. Since there is no technical change labor-values are constant.

The real wage is a bundle of commodities used-up in the reproduction of the working population. But the labor-value of the real wage does not reappear in the 'labor-value' of the commodity labor-power. A unit of labor-time, for example 1 worker-hour, by definition has no connection to the composition or scale of the real wage (see also Wright (2009, Sec. 6)). In consequence, worker households are a *source* of labor-time, since they supply labor during the production of commodities, and also a *sink* of labor-time, since they consume a real wage that costs a definite amount of labor-time to replace.

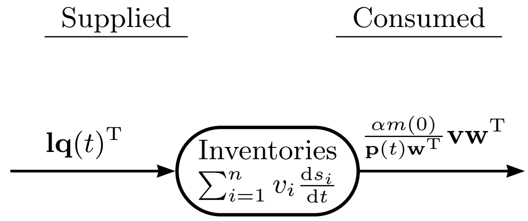


FIG. 4. The aggregate demand for labor. Any imbalances between the labor supplied to production and the labor-value consumed (in the form of the real wage) causes a corresponding change in the stock of stored inventories.

The aggregate demand for labor-time is the labor-value of the real wage. But the current supply of labor-time may not match this demand. Figure 4 depicts the relationship between the aggregate demand and supply of labor-time. Workers supply $\mathbf{lq}^{\mathrm{T}}$ labor-time to production. This direct labor is combined with indirect labor-value in the form of other circulating commodities that get used-up in the production process. The used-up inputs are replaced and a net product, the real wage, is output. The consumption of the real wage represents a cost of $(\alpha m(0)/\mathbf{pw}^{\mathrm{T}})\mathbf{vw}^{\mathrm{T}}$ labor-time. If the amount of labor-time supplied is greater than (resp. less than) the amount of labor-value consumed then the total labor-value of the inventory stock, $\sum v_i \frac{\mathrm{d}s_i}{\mathrm{d}t}$, increases (resp. decreases) by an equal amount; that is, if the aggregate demand for labor is less than (resp. more than) its supply then inventories start to grow (resp. shrink). This relation is summarized by the following aggregate labor demand equation.

Lemma 4. The change in the labor-value of inventories equals the net supply of labor-value from worker households,

$$\sum_{i=1}^{n} v_i \frac{\mathrm{d}s_i}{\mathrm{d}t} = \mathbf{lq}^{\mathrm{T}} - \frac{\alpha m(0)}{\mathbf{pw}^{\mathrm{T}}} \mathbf{vw}^{\mathrm{T}}. \tag{12}$$

Proof. From equation (4) we get

$$\begin{aligned} \sum_{i=1}^{n} v_i \frac{\mathrm{d}s_i}{\mathrm{d}t} &= \sum_{i=1}^{n} \left(v_i q_i - v_i \mathbf{A}_{(i)} \mathbf{q}^{\mathrm{T}} - \frac{\alpha m(0)}{\mathbf{pw}^{\mathrm{T}}} v_i w_i \right) \\ &= \mathbf{vq}^{\mathrm{T}} - \mathbf{vAq}^{\mathrm{T}} - \frac{\alpha m(0)}{\mathbf{pw}^{\mathrm{T}}} \mathbf{vw}^{\mathrm{T}}. \end{aligned} \tag{13}$$

From labor-value equation (11) infer that $\mathbf{vq}^{\mathrm{T}} = \mathbf{vAq}^{\mathrm{T}} + \mathbf{lq}^{\mathrm{T}}$. Substitute into equation (13) and the conclusion follows. $\square$

The aggregate labor demand equation is 'dual' to the money conservation identity. By thinking in terms of labor-values we can abstract from different commodity types and represent the economy as a complex network of flows of two substances, money and labor-value. Money flows in the opposite direction to labor-value and, as we shall see, functions as a feedback mechanism that eliminates mismatches between the labor-time supplied to different sectors of production and the final demand for different commodity types.

The total stock of money held in the form of bank deposits is conserved. The total stock of labor-value is the total labor-value of stored inventories. But this stock is not conserved; it alters with the change in the level of employment and the labor-value of the real wage. Aggregate labor demand equation (12) also plays an important role in proving the convergence of market prices to natural prices.

VIII. THE SIMPLE PRODUCTION SYSTEM

The simple production system has $2n$ time-dependent variables: (i) $\mathbf{p}(t) = [p_i(t)]$, a $n \times 1$ price vector, and (ii) $\mathbf{q}(t) = [q_i(t)]$, a $n \times 1$ quantity vector; $n^2 + 3n + 5$ constant parameters: (i) $\mathbf{A} = [a_{i,j}]$, a $n \times n$ input-output matrix, (ii) $\mathbf{l} = [l_i]$, a $n \times 1$ labor coefficient vector, (iii) $\mathbf{w} = [w_i]$, $n \times 1$ real wage vector, (iv) L, the total available labor force, (v) $m(0)$, worker savings, (vi) α, workers' propensity to consume, and (vii) $n + 2$ adjustment coefficients $\lambda_1, \ldots, \lambda_n, \lambda_q, \lambda_w$; and $3n$ initial values, consisting of initial prices, $\mathbf{p}(0) = [p_i(0)] >> \mathbf{0}$, initial quantities, $\mathbf{q}(0) = [q_i(0)] >> \mathbf{0}$, and initial inventory levels, $\mathbf{s}(0) = [s_i(0)] >> \mathbf{0}$, subject to the restriction that the initial level of employment is less than the total available labor force, $\mathbf{lq}(0)^{\mathrm{T}} < L$. The wage rate, inventory levels and worker savings are eliminated. The simple production system is therefore a $2n$-dimensional nonlinear system of ordinary differential equations in prices and quantities,

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{f}(\mathbf{x}(t)) \\ &= \left[\ \tfrac{\mathrm{d}p_1}{\mathrm{d}t}, \ \ldots, \ \tfrac{\mathrm{d}p_n}{\mathrm{d}t}, \ \tfrac{\mathrm{d}q_1}{\mathrm{d}t}, \ \ldots, \ \tfrac{\mathrm{d}q_n}{\mathrm{d}t} \ \right]^{\mathrm{T}},\end{aligned} \tag{14}$$

where

$$\begin{aligned}\frac{\mathrm{d}p_i}{\mathrm{d}t} &= -\frac{\lambda_i}{p_i(0)^{1/\lambda_i} s_i(0)} p_i^{1+\lambda_i} \left(q_i - \mathbf{A}_{(i)}\mathbf{q}^{\mathrm{T}} - \frac{\alpha m}{\mathbf{p}\mathbf{w}^{\mathrm{T}}} w_i \right) \\ \frac{\mathrm{d}q_i}{\mathrm{d}t} &= \frac{\lambda_q}{l_i k_w}(L - \mathbf{lq}^{\mathrm{T}})^{\lambda_w} (p_i(\mathbf{A}_{(i)}\mathbf{q}^{\mathrm{T}} + \frac{\alpha m(0)}{\mathbf{p}\mathbf{w}^{\mathrm{T}}} w_i) - \\ &\qquad q_i(\mathbf{p}\mathbf{A}^{(i)} + l_i k_w \frac{1}{(L - \mathbf{lq}^{\mathrm{T}})^{\lambda_w}})).\end{aligned}$$

Next we will characterize the equilibrium of the system.

IX. THE EQUILIBRIUM STATE

The economy has reached an equilibrium point when prices and quantities are constant, that is $\frac{\mathrm{d}p_i}{\mathrm{d}t} = \frac{\mathrm{d}q_i}{\mathrm{d}t} = 0$ for all i. First we specify the equilibrium level of employment and the wage rate; then we derive expressions for equilibrium prices and quantities. We restrict the domain of analysis to the positive orthant in order to ignore uninteresting equilibria, such as zero prices and quantities, $\mathbf{p} = \mathbf{q} = \mathbf{0}$.

IX.1. Zero profits in equilibrium

Quantities adjust according to profit signals, which indicate imbalances between costs and revenues within each sector. Since quantities are constant in equilibrium there is not an incentive to reallocate capital; therefore profits are zero.

Proposition 1. Profits are zero in equilibrium, $\pi_i = 0$ for all i.

Proof. Substituting $\frac{\mathrm{d}q_i}{\mathrm{d}t} = 0$ into quantity adjustment equation (6) implies $\pi_i = 0$ for all i. □

IX.2. The equilibrium wage bill equals aggregate demand

Money conservation identity (10) implies that the aggregate demand always return either in the form of wage income or profits. Since profits are zero in equilibrium the total wage income equals the aggregate demand.

Proposition 2. The equilibrium total wage bill equals aggregate demand,

$$\mathbf{lq}^{*\mathrm{T}} w^* = \alpha m(0). \tag{15}$$

Proof. By Lemma 3, $\alpha m^* = \alpha m(0)$. In equilibrium $\frac{\mathrm{d}m}{\mathrm{d}t} = 0$ and by Proposition 1, $\pi_i = 0$ for all i. Substitute these equilibrium conditions into equation (2). □

In equilibrium, therefore, workers spend all their wage income on consumption goods.

IX.3. An unemployment equilibrium

Cross-dual adjustment tends to eliminate arbitrage opportunities (e.g., profit differentials) but it lacks any tendency to fully employ all labor resources.

Proposition 3. Equilibrium employment, θ^*, is implicitly defined by

$$\theta^* = \frac{\alpha m(0)}{k_w}(L - \theta^*)^{\lambda_w}, \tag{16}$$

where $\theta^* = \mathbf{lq}^{*\mathrm{T}}$. In general equation (16) cannot be solved algebraically for arbitrary λ_w.

Proof. Equation (15) implies

$$\mathbf{lq}^{*\mathrm{T}} = \frac{\alpha m(0)}{w^*}. \tag{17}$$

Lemma 1 yields equilibrium wages in terms of equilibrium employment,

$$w^* = k_w \frac{1}{(L - \mathbf{lq}^{*\mathrm{T}})^{\lambda_w}}. \tag{18}$$

Substitute (18) into (17) and rearrange to yield the conclusion. □

An immediate consequence is that the equilibrium level of employment cannot be full employment.

Lemma 5. The equilibrium level of employment is positive but cannot be full employment; that is, $\theta^* > 0$ and $\theta^* \neq L$.

Proof. (i) Assume $\theta^* \leq 0$ then the left-hand-side of (16) is non-positive and, since all constants are positive, the right-hand-side of (16) is positive, which is a contradiction; hence it cannot be the case that $\theta^* \leq 0$. (ii) Assume $\theta^* = L$ then the left-hand-side of (16) is positive but the right-hand-size is zero, which is a contradiction; hence it cannot be the case that $\theta^* = L$. □

We can briefly examine what determines the equilibrium level of employment by writing (16) in full,

$$\theta^* = \frac{\alpha m(0)}{w(0)} \left(\frac{L - \theta^*}{L - \mathbf{l}\mathbf{q}(0)^{\mathrm{T}}} \right)^{\lambda_w}.$$

High aggregate demand, $\alpha m(0)$, relative to the initial wage rate, $w(0)$, contributes to a high equilibrium level of employment, θ^*, given an initial level of employment, $\mathbf{l}\mathbf{q}(0)^{\mathrm{T}}$ and wage adjustment speed, λ_w. So low wages combined with high aggregate demand (a possible combination since profit is distributed to workers) tends to increase the equilibrium level of employment. The model, therefore, has a Keynesian flavor: aggregate demand drives the economy, money and adjustment speeds have real effects, and market adjustment, by itself, does not guarantee full employment. Labor is efficiently allocated between the different sectors of production in equilibrium but the economy is not operating at full capacity.

IX.4. Equilibrium wage rate

The equilibrium wage rate is simply aggregate demand divided by the equilibrium level of employment.

Proposition 4. The equilibrium wage rate is

$$w^* = \frac{\alpha m(0)}{\theta^*}. \tag{19}$$

Proof. This follows directly from equation (17) and Lemma 3. □

IX.5. Equilibrium prices

We can now derive an expression for equilibrium prices.

Proposition 5. Absolute equilibrium prices are proportional to labor-values,

$$\mathbf{p}^* = \mathbf{v}w^*, \tag{20}$$

where the constant of proportionality is the equilibrium wage rate, w^*.

Proof. In equilibrium $\pi_i = 0$ for all i. So equations (1) imply

$$p_i^* \left(\mathbf{A}_{(i)}\mathbf{q}^{*\mathrm{T}} + w_i \frac{\alpha m(0)}{\mathbf{p}^*\mathbf{w}^{\mathrm{T}}} \right) = q_i^*(\mathbf{p}^*\mathbf{A}^{(i)} + l_i w^*) \tag{21}$$

for all i. Write equations (21) in vector form and use equation (15) (i.e., $\alpha m(0) = \mathbf{l}\mathbf{q}^{*\mathrm{T}}w^*$) to replace aggregate demand with the total wage bill to give

$$\mathbf{p}^* \left(\mathbf{A} + \frac{w^*}{\mathbf{p}^*\mathbf{w}^{\mathrm{T}}}\mathbf{w}^{\mathrm{T}}\mathbf{l} \right) \mathbf{q}^{*\mathrm{T}} = (\mathbf{p}^*\mathbf{A} + \mathbf{l}w^*)\mathbf{q}^{*\mathrm{T}}. \tag{22}$$

In equilibrium $\frac{\mathrm{d}p_i}{\mathrm{d}t} = 0$ and therefore price adjustment equation (5) implies $\frac{\mathrm{d}s_i}{\mathrm{d}t} = 0$. Substituting $\frac{\mathrm{d}s_i}{\mathrm{d}t} = 0$ into equations (4) gives

$$q_i^* = \mathbf{A}_{(i)}\mathbf{q}^{*\mathrm{T}} + \frac{\alpha m(0)}{\mathbf{p}^*\mathbf{w}^{\mathrm{T}}} w_i \tag{23}$$

for all i. Substitute (15) into equations (23) to give

$$\mathbf{q}^{*\mathrm{T}} = \left(\mathbf{A} + \frac{w^*}{\mathbf{p}^*\mathbf{w}^{\mathrm{T}}}\mathbf{w}^{\mathrm{T}}\mathbf{l} \right) \mathbf{q}^{*\mathrm{T}}. \tag{24}$$

Use (24) to simplify the left-hand-side of (22) to yield

$$\mathbf{p}^*\mathbf{q}^{*\mathrm{T}} = (\mathbf{p}^*\mathbf{A} + \mathbf{l}w^*)\mathbf{q}^{*\mathrm{T}}. \tag{25}$$

Rearranging,

$$(\mathbf{p}^* - \mathbf{p}^*\mathbf{A} - \mathbf{l}w^*)\mathbf{q}^{*\mathrm{T}} = 0, \tag{26}$$

which has the form of a dot product equation, $\mathbf{x} \cdot \mathbf{y} = 0$, with $\mathbf{x} = \mathbf{p}^* - \mathbf{p}^*\mathbf{A} - \mathbf{l}w^*$ and $\mathbf{y} = \mathbf{q}^*$. Since $\mathbf{x}$ is not orthogonal to $\mathbf{y}$ and $\mathbf{y} \neq \mathbf{0}$ then

$$\mathbf{p}^* = \mathbf{p}^*\mathbf{A} + \mathbf{l}w^*. \tag{27}$$

So $\mathbf{p}^* = \mathbf{l}(\mathbf{I} - \mathbf{A})^{-1}w^* = \mathbf{v}w^*$. □

An equivalent expression for equilibrium prices is

$$\mathbf{p}^* = \mathbf{p}^*\mathbf{A} + \mathbf{l}w^*$$

(see the proof of Proposition 5 in the appendix), which is identical to the linear production (or input-output) model of prices under conditions of simple production (e.g., see Pasinetti (1977, Ch. 5)). The dynamical system embeds the linear production price model as a special case at equilibrium.

In linear production models we deduce *relative* prices since the system of simultaneous equations is able to fix prices only up to an arbitrary choice of *numéraire*. Sometimes this side-effect of working with static models is presented as a necessary feature of economic analysis. But this dynamic model, in which money stocks and flows are explicit, determines *absolute* prices. The equilibrium price level is not arbitrary but completely determined by the initial conditions and laws of motion of the economy.

Equilibrium prices are those that ensure zero profits everywhere. The sum of input costs ($\mathbf{p}^*\mathbf{A}$) and wage costs ($\mathbf{l}w^*$) per unit output are exactly equal to unit revenues ($\mathbf{p}$). But we can dig deeper. Earlier we interpreted labor-value equation (11) as a sum of 'dead' labor embodied in input commodities plus an addition of 'living' labor. This interpretation suggests a *sequence* of phases in which input commodities arrive at the locus of production to be transformed by work into output commodities of equivalent labor-value. But we can also interpret labor-values in a *parallel* fashion, as intimated by Marx in the following quotation.

> [Raw] cotton, yarn, fabric, are not only produced one after the other and from one another, but they are produced and reproduced *simultaneously*, alongside one another. What appears

> as the effect of antecedent labor, if one considers the production process of the individual commodity, presents itself at the same time as the effect of coexisting labor, if one considers the *reproduction process* of the commodity, that is, if one considers this production process in its continuous motion and in the entirety of its conditions, and not merely an isolated action or a limited part of it. There exists not only a cycle comprising various phases, but all the phases of the commodity are simultaneously produced in the various spheres and branches of production. If the same peasant just plants flax, then spins it, then weaves it, these operations are performed in succession, but not simultaneously as the mode of production based on the division of labor within society presupposes. (Marx, 2000)

Perelman (1987) describes how Marx adopted the concept of coexisting labor from Thomas Hodgskin. Let's follow Marx's logic, translating into the terms of the model. The total workforce is split-up into groups employed in different sectors of production. Each group works in parallel to produce commodities either consumed or used-up as intermediate inputs. The production of a unit of commodity i requires direct labor l_i plus input commodities $\mathbf{A}^{(i)}$. During this production the input commodities used-up are simultaneously replaced by direct labor $\mathbf{l}\mathbf{A}^{(i)}$ operating in parallel in other sectors of the economy. But this simultaneous production *itself* uses-up input commodities $\mathbf{A}\mathbf{A}^{(i)}$, which are also simultaneously replaced with the expenditure of additional direct labor $\mathbf{l}\mathbf{A}\mathbf{A}^{(i)}$ operating in parallel. To count all the 'coexisting labor', λ_i, that works in parallel to contribute to the production of a unit of commodity i we must continue the sum; that is,

$$\begin{aligned}\lambda_i &= l_i + \mathbf{l}\mathbf{A}^{(i)} + \mathbf{l}\mathbf{A}\mathbf{A}^{(i)} + \mathbf{l}\mathbf{A}^2\mathbf{A}^{(i)} + \dots \\ &= l_i + \mathbf{l}(\mathbf{I} + \mathbf{A} + \mathbf{A}^2 + \dots)\mathbf{A}^{(i)} \\ &= l_i + \mathbf{l}(\sum_{n=0}^{\infty}\mathbf{A}^n)\mathbf{A}^{(i)}.\end{aligned}$$

So the vector λ of 'coexisting labor' required to reproduce a unit bundle $\mathbf{u} = [1]$ of commodities is

$$\begin{aligned}\lambda &= \mathbf{l} + \mathbf{l}(\sum_{n=0}^{\infty}\mathbf{A}^n)\mathbf{A} \\ &= \mathbf{l}\sum_{n=0}^{\infty}\mathbf{A}^n.\end{aligned} \tag{28}$$

An alternative representation of the infinite series in equation (28) is the Leontief inverse $(\mathbf{I} - \mathbf{A})^{-1}$; hence, by equation (11),

$$\lambda = \mathbf{l}(\mathbf{I} - \mathbf{A})^{-1} = \mathbf{v}$$

and the total amount of 'coexisting labor' required to reproduce a unit of commodity i is equal to its labor-value. A labor-value therefore is simply the *total amount of coexisting labor required to reproduce a commodity*, where the coexisting labor works in parallel to produce a unit of output and also replace the used-up means of production. In this interpretation the concept of 'dead' labor 'embodied' in the form of means of production has no role.

The coexisting labor working in parallel to reproduce a commodity is identical to Pasinetti's concept of a 'vertically integrated' sector (Pasinetti, 1980). A vertically integrated sector is a conceptual classification of the economy that cuts across the 'horizontal' boundaries of work location and firm ownership.

The natural price of a commodity is therefore the wages of the total coexisting labor required to reproduce it. So commodities that require more of society's labor-time to produce sell at higher prices in equilibrium.

IX.6. Positive inventories in equilibrium

Stored inventories do not decay. And firms do not attempt to reduce their absolute level of inventory but only modify their prices in response to relative changes in their level of inventory. In consequence, inventories are positive in equilibrium.

Lemma 6. Equilibrium prices, $\mathbf{p}^*$, and equilibrium inventories, $\mathbf{s}^*$, are positive.

Proof. Since $\theta^* \in (0, L)$ (Lemma 5) then Proposition 4 implies $w^* > 0$. Labor-values $\mathbf{v}$ are positive. So $\mathbf{p}^* = \mathbf{v}w^*$ (Theorem 5) implies prices are positive. Inventories are a sign preserving function of prices (Lemma 2); hence inventories are also positive. □

IX.7. Equilibrium quantities

Proposition 6. Equilibrium quantities are

$$\mathbf{q}^* = \mathbf{q}^*\mathbf{A}^{\mathrm{T}} + \mathbf{w}^* \tag{29}$$

where

$$\mathbf{w}^* = \frac{\theta^*}{\mathbf{v}\mathbf{w}^{\mathrm{T}}}\mathbf{w} \tag{30}$$

is the equilibrium real wage.

Proof. $\frac{\mathrm{d}s_i}{\mathrm{d}t} = 0$ for all i in equilibrium so equations (4) imply

$$q_i^* = \mathbf{A}_{(i)}\mathbf{q}^{*\mathrm{T}} + \frac{\alpha m(0)}{\mathbf{p}^*\mathbf{w}^{\mathrm{T}}}w_i \tag{31}$$

for all i. Write equations (31) in vector form and use equation (15) (i.e., $\alpha m(0) = \mathbf{l}\mathbf{q}^{*\mathrm{T}}w^*$) to replace aggregate demand to give

$$\mathbf{q}^* = \mathbf{q}^*\mathbf{A}^{\mathrm{T}} + \frac{\mathbf{l}\mathbf{q}^{*\mathrm{T}}w^*}{\mathbf{p}^*\mathbf{w}^{\mathrm{T}}}\mathbf{w} \tag{32}$$

By Proposition 3 equilibrium employment, $\theta^* = \mathbf{l}\mathbf{q}^{*\mathrm{T}}$, is determined entirely by the initial conditions. So the equilibrium real wage

$$\mathbf{w}^* = \frac{\theta^* w^*}{\mathbf{p}^*\mathbf{w}^{\mathrm{T}}}\mathbf{w}$$

is simply $\mathbf{w}$ scaled by the number of units that may be purchased by the equilibrium total money wage (i.e., equilibrium consumption demand). By Proposition 5, $\mathbf{w}^* = (\theta^*/\mathbf{v}\mathbf{w}^{\mathrm{T}})\mathbf{w}$. □

Again, this expression for equilibrium quantities is identical to the linear production solution under conditions of simple production (e.g., see the discussion of the Open Leontief system in Pasinetti (1977, Ch. 4)). The equilibrium scale of production consists of the collection of commodities used-up as means of production, $\mathbf{q}^*\mathbf{A}^{\mathrm{T}}$, and the net product, $\mathbf{w}^*$, which is the real wage consumed by worker households. So this dynamic system embeds the complete linear production model of simple production at its equilibrium point.

Equation (30), which defines the equilibrium real wage, has an interesting interpretation. It implies that the labor-value of the real wage equals the labor employed,

$$\mathbf{v}\mathbf{w}^{*\mathrm{T}} = \theta^*. \tag{33}$$

So in equilibrium the ratio of the direct labor supplied to production, θ^*, to the labor-embodied in the real wage, $\mathbf{v}\mathbf{w}^{*\mathrm{T}}$, is 1. In this sense the economy is efficient: all the labor supplied to production returns in the form of consumption goods. No labor is 'lost' due to quantity imbalances. Or, equivalently, equation (33) states that the total coexisting labor required to reproduce the real wage, $\mathbf{v}\mathbf{w}^{*\mathrm{T}}$, equals the total labor employed, θ^*. In the context of an equilibrium model Pasinetti (1980) interprets (33) as expressing two different ways of classifying, or dis-aggregating, the total employed labor force. The expression $\mathbf{l}\mathbf{q}^{\mathrm{T}}$ classifies the total labor 'according to the criterion of the industry in which [it is] required'. The expression $\mathbf{v}\mathbf{w}^{*\mathrm{T}}$ classifies the total labor 'according to the criterion of the vertically integrated sector for which [it is] directly and indirectly required'. In equilibrium there is no contradiction between the horizontal and vertical allocation of the labor force.

IX.8. The independence of 'long-period' equilibrium from initial endowments and prices

The price and quantity equilibrium $[\mathbf{p}^*, \mathbf{q}^*]$ (defined by equations (5) and (6) respectively) is independent of initial prices $\mathbf{p}(0)$ and initial inventory levels $\mathbf{s}(0)$. The 'long-period' equilibrium of the economy is therefore unaffected by an arbitrary change to initial prices or inventory levels. For example, the equilibrium price and quantity of a particular commodity i, say corn, is the same, all other things being equal, regardless of whether corn initially has a high price ($p_i(0) >> 0$) and is scarce ($s_i(0) \approx 0$), or initially has a low price ($p_i(0) \approx 0$) and is abundant ($s_i(0) >> 0$). In this sense, non-equilibrium prices, such as initial market prices, are accidental costs, determined by transient supply and demand imbalances. Equilibrium prices, in contrast, are necessary costs, determined by the objective costs of production.

X. STABILITY OF NATURAL PRICES

The mere existence of a natural price equilibrium does not imply that the economy will gravitate towards it. We will now prove that the equilibrium of the simple production system is in fact at least locally asymptotically stable. So any trajectories that begin in the domain of attraction converge to the natural price equilibrium and stay there. In this model the process of cross-dual adjustment converges – just as the classical authors expected.

The stability proof is somewhat opaque, so in addition to providing some intuition behind the proof we will also try to extract its economic meaning. But readers content with the conclusion can skip this section.

X.1. The vector Lyapunov function approach

Analytical solutions to systems of nonlinear differential equations can be very difficult to obtain. Fortunately we can analyze stability without solving the equations. Lyapunov's direct method (e.g. see Brauer and Nohel (1989)) states that if a special kind of function can be found, which summarizes the system state and decreases monotonically along system trajectories until it reaches zero at the equilibrium point, then the system is stable. In many physical models we can interpret the Lyapunov function as the total energy of the system. For example, a dissipative system, such as a ball rolling in a bowl subject to friction, will lose energy over time until it reaches zero, at which point the ball is at rest. Unfortunately, this method can be challenging to apply because there are no general methods for constructing a suitable Lyapunov function.

A generalization of Lyapunov's method, called the vector Lyapunov approach, can sometimes provide an alternative route. Rather than summarizing the state of the dynamical system by a single function that we prove must decrease monotonically over time, we instead summarize the state by q scalar functions with gradients that we prove are bounded by a new, q-dimensional dynamical system that is stable. Each component of the vector Lyapunov function is required to satisfy less stringent conditions, which gives more flexibility when searching for candidate functions.

In many cases, high dimensional dynamic systems often suggest natural decompositions. The vector Lyapunov approach can exploit this structure by specializing each part of the vector function to each subsystem.

First, some notation. The gradient of a scalar function $f(x_1, x_2, \ldots, x_n)$ of n variables is the $1 \times n$ vector

$$\nabla f(\mathbf{x}) = \left[\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n}\right],$$

which is formed by partially differentiating f with respect to each of its n variables. We can generalize this concept to vectors of functions. The gradient of a q-dimensional vector function $\mathbf{f}$ of n variables,

$$\mathbf{f}(\mathbf{x}) = [f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_q(\mathbf{x})]^{\mathrm{T}},$$

is the $q \times n$ matrix

$$\nabla \mathbf{f}(\mathbf{x}) = [\nabla f_1(\mathbf{x}), \nabla f_2(\mathbf{x}), \ldots, \nabla f_q(\mathbf{x})]^{\mathrm{T}}.$$

Consider the $2n$-dimensional simple production system, $\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t))$, defined by equation (14), and a candidate q-dimensional ($q < 2n$) vector Lyapunov function, $\mathbf{V}(\mathbf{x}) = [\mathbf{V}_1(\mathbf{x}), \mathbf{V}_2(\mathbf{x}), \ldots, \mathbf{V}_q(\mathbf{x})]^{\mathrm{T}}$. The gradient of $\mathbf{V}$ along system trajectories is the $q \times 1$ vector

$$\nabla \mathbf{V}(\mathbf{x})\mathbf{f}(\mathbf{x}) = [\nabla f_1(\mathbf{x})\mathbf{f}(\mathbf{x}), \nabla f_2(\mathbf{x})\mathbf{f}(\mathbf{x}), \ldots, \nabla f_q(\mathbf{x})\mathbf{f}(\mathbf{x})]^{\mathrm{T}}.$$

To prove stability using a vector Lyapunov function $\mathbf{V}(\mathbf{x})$ we need to show that its gradient on system trajectories is bounded by a vector function $\mathbf{w}$ of $\mathbf{V}$ that satisfies certain special properties. The key theorem is:

Theorem 7. Consider the nonlinear dynamical system

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t)), \tag{34}$$

where $\mathbf{x}^*$ is an equilibrium such that $\mathbf{f}(\mathbf{x}^*) = \mathbf{0}$. Assume there exists a continuously differentiable vector function $\mathbf{V} : \mathcal{D} \to \overline{\mathbb{R}}_+^q$, where $\overline{\mathbb{R}}_+^q$ is the set of q-dimensional non-negative real vectors, and a positive vector $\mathbf{u} \in \mathbb{R}_+^q$ such that $\mathbf{V}(\mathbf{x}^*) = \mathbf{0}$, the scalar function $v : \mathcal{D} \to \overline{\mathbb{R}}_+$ defined by $v(\mathbf{x}) = \mathbf{u}\mathbf{V}(x)$, $x \in \mathcal{D}$, is such that $v(\mathbf{x}) > 0$, $\mathbf{x} \neq \mathbf{x}^*$, and

$$\nabla \mathbf{V}(\mathbf{x})\mathbf{f}(\mathbf{x}) \leq\leq \mathbf{w}(\mathbf{V}(\mathbf{x})),$$

$\mathbf{x} \in \mathcal{D}$, where $\mathbf{w} : \mathbb{R}^q \to \mathbb{R}^q$ is continuous, satisfies the Kamke condition, and $\mathbf{w}(\mathbf{0}) = \mathbf{0}$. Then if the zero solution $\mathbf{z}(t) = \mathbf{0}$ to the comparison system

$$\dot{\mathbf{z}}(t) = \mathbf{w}(\mathbf{z}(t))$$

is asymptotically stable then the equilibrium solution $\mathbf{x}(t) = \mathbf{x}^*$ to (34) is asymptotically stable.

Proof. See Haddad and Chellaboina (2008, pg. 304). □

The idea behind this theorem is that we can conclude stability if the trajectory of the gradient of the vector Lyapunov function is always 'contained within' a new, lower-dimensional dynamic system, called the comparison system, which is stable and converges to zero. Hence trajectories that start in a local domain of attraction must converge to the equilibrium point.

The comparison system must satisfy some technical conditions, in particular the Kamke condition:

Definition 1. A function $\mathbf{w} = [w_1, \ldots, w_q] : \mathbb{R}^q \to \mathbb{R}^q$ satisfies the *Kamke condition* if $w_i(\mathbf{x}') \leq w_i(\mathbf{x}'')$, $i = 1, \ldots, q$ for all $\mathbf{x}', \mathbf{x}'' \in \mathbb{R}^q$ such that $\mathbf{x}'_j \leq \mathbf{x}''_j$, $\mathbf{x}'_i = \mathbf{x}''_i$, $j = 1, \ldots, q$, $i \neq j$, where $\mathbf{x}_i$ denotes the ith component of $\mathbf{x}$.

We will prove stability by constructing a vector Lyapunov function that satisfies the conditions of Theorem 7.

X.2. A candidate vector Lyapunov function

Consider the two-dimensional vector Lyapunov function candidate $\mathbf{V} : \overline{\mathbb{R}}_+^{2n} \to \overline{\mathbb{R}}_+^2$,

$$\begin{aligned}\mathbf{V}([\mathbf{p}, \mathbf{q}]) &= [V_1([\mathbf{p}, \mathbf{q}]), V_2([\mathbf{p}, \mathbf{q}])]^{\mathrm{T}} \\ &= \left[\left(\frac{1}{w} \sum_{i=1}^{n} \pi_i \right)^2, V_1 \sum_{i=1}^{n} \frac{1}{\lambda_i^2} v_i s_i \right]^{\mathrm{T}},\end{aligned} \tag{35}$$

defined on the domain $\mathcal{D} = \{[\mathbf{p}, \mathbf{q}] \in \mathbb{R}_+^{2n} : \mathbf{l}\mathbf{q} \leq L\}$ (i.e., prices $\mathbf{p}$ and quantities $\mathbf{q}$ are positive, and total employment never exceeds the total labor force). $\mathbf{V}$ summarizes the state of the simple production system in terms of two scalar functions, V_1 and V_2.

V_1 is the square of the total profit divided by the wage rate. The money conservation identity (10) and wage trajectory (7) imply that

$$\frac{1}{w} \sum_{i=1}^{n} \pi_i = \frac{\alpha m(0)}{w} - \mathbf{l}\mathbf{q}^{\mathrm{T}}.$$

So V_1 can also be interpreted as the square of the difference between total labor commanded by aggregate demand, $\alpha m(0)/w$, and the total labor supplied to production, $\mathbf{l}\mathbf{q}^{\mathrm{T}}$. V_1 is a summary function of only the quantity part of the system state.

V_2 equals V_1 multiplied by a transformed sum of the labor-value of the stock of inventories. The labor-value of each sector's inventory is transformed by a simple function of the respective price adjustment coefficient. V_2 is a summary function of the total system state.

At equilibrium $\mathbf{V}(\mathbf{p}^*, \mathbf{q}^*) = [0, 0]$, since $\pi_i = 0$ for all i. The scalar function $\mathbf{u}\mathbf{V}([\mathbf{p}, \mathbf{q}])$, where $\mathbf{u} = [1, 1]$, is positive definite; that is $\mathbf{u}\mathbf{V}([\mathbf{p}, \mathbf{q}]) > 0$ for $\mathbf{p} \neq \mathbf{p}^*$ and $\mathbf{q} \neq \mathbf{q}^*$.

The gradient of $\mathbf{V}$ along system trajectories is $\nabla \mathbf{V}\mathbf{f} = [\nabla V_1 \mathbf{f}, \nabla V_2 \mathbf{f}]$. We will examine each component of the gradient in turn and show that it is bounded by a function of the candidate vector Lyapunov function.

Lemma 8. The gradient of V_1 on system trajectories is strictly decreasing.

$$\begin{aligned}\nabla V_1 \mathbf{f} &\leq -k_\alpha V_1 \\ &\leq 0,\end{aligned}$$

where $k_\alpha = 2\lambda_q(\lambda_w \frac{\alpha m(0)}{k_w} L^{\lambda_w - 1} + 1)$ is a positive constant; i.e., the square of the difference between aggregate labor-commanded and total employment monotonically decreases along system trajectories.

Proof.

$$\begin{aligned}\nabla V_1 \mathbf{f} &= \sum_{i=1}^{n} \frac{\partial V_1}{\partial p_i} \frac{\mathrm{d}p_i}{\mathrm{d}t} + \sum_{i=1}^{n} \frac{\partial V_1}{\partial q_i} \frac{\mathrm{d}q_i}{\mathrm{d}t} \\ &= -\sum_{i=1}^{n} 2 \left(\frac{1}{w} \sum_{j=1}^{n} \pi_j \right) \\ &\quad (\lambda_w \frac{\alpha m(0)}{k_w} (L - \mathbf{l}\mathbf{q}^{\mathrm{T}})^{\lambda_w - 1} + 1) \lambda_q \frac{1}{w} \pi_i \\ &= -2\lambda_q (\lambda_w \frac{\alpha m(0)}{k_w} (L - \mathbf{l}\mathbf{q}^{\mathrm{T}})^{\lambda_w - 1} + 1) \left(\frac{1}{w} \sum_{i=1}^{n} \pi_i \right)^2 \\ &= -2\lambda_q (\lambda_w \frac{\alpha m(0)}{k_w} (L - \mathbf{l}\mathbf{q}^{\mathrm{T}})^{\lambda_w - 1} + 1) V_1 \\ &\leq -2\lambda_q (\lambda_w \frac{\alpha m(0)}{k_w} L^{\lambda_w - 1} + 1) V_1 \\ &\leq -k_\alpha V_1 \\ &\leq 0.\end{aligned}$$

□

Lemma 9. The gradient of V_2 on system trajectories is bounded by a linear function of V_1 and V_2,

$$\nabla V_2 \mathbf{f} < L V_1 - k_\alpha V_2. \tag{36}$$

Proof.

$$\nabla V_2 \mathbf{f} = \sum_{i=1}^{n} \frac{\partial V_2}{\partial p_i}\frac{\mathrm{d}p_i}{\mathrm{d}t} + \sum_{i=1}^{n} \frac{\partial V_2}{\partial q_i}\frac{\mathrm{d}q_i}{\mathrm{d}t}$$
$$= \sum_{i=1}^{n} \left((V_1 \frac{\partial}{\partial p_i} \sum_{j=1}^{n} \frac{1}{\lambda_j^2} v_j s_j + \frac{\partial V_1}{\partial p_i} \sum_{j=1}^{n} \frac{1}{\lambda_j^2} v_j s_j) \frac{\mathrm{d}p_i}{\mathrm{d}t} \right)$$
$$+ \sum_{i=1}^{n} \frac{\partial V_1}{\partial q_i}\frac{\mathrm{d}q_i}{\mathrm{d}t} \sum_{j=1}^{n} \frac{1}{\lambda_j^2} v_j s_j$$
$$= \sum_{i=1}^{n} (V_1 \frac{\partial}{\partial p_i} \sum_{j=1}^{n} \frac{1}{\lambda_i^2} \frac{v_j p_j(0)^{\frac{1}{\lambda_j}} s_j(0)}{p_j^{\lambda_j}}) \frac{\mathrm{d}p_i}{\mathrm{d}t}$$
$$-k_\alpha V_1 \sum_{j=1}^{n} \frac{1}{\lambda_j^2} v_j s_j$$
$$= -\sum_{i=1}^{n} V_1 \frac{v_i p_i(0)^{\frac{1}{\lambda_i}} s_i(0)}{\lambda_i p_i^{\lambda_i+1}} \frac{\mathrm{d}p_i}{\mathrm{d}t} - k_\alpha V_2$$
$$= V_1 \sum_{i=1}^{n} v_i \frac{\mathrm{d}s_i}{\mathrm{d}t} - k_\alpha V_2.$$

Prices $\mathbf{p}$ are always positive and therefore aggregate labor demand equation (12) implies

$$\sum_{i=1}^{n} v_i \frac{\mathrm{d}s_i}{dt} = \mathbf{l}\mathbf{q}^{\mathrm{T}} - (\alpha m(0)\mathbf{v}\mathbf{w}^{\mathrm{T}}) \frac{1}{\mathbf{p}\mathbf{w}^{\mathrm{T}}}$$
$$< \mathbf{l}\mathbf{q}^{\mathrm{T}}$$
$$< L.$$

Hence,

$$\nabla V_2 \mathbf{f} < L V_1 - k_\alpha V_2.$$

□

Lemmas 8 and 9 define differential inequalities that will allow us to construct a comparison system.

X.3. Stability of the comparison system

Since $\nabla V_1 \mathbf{f} \leq -k_\alpha V_1$ and $\nabla V_2 \mathbf{f} < L V_1 - k_\alpha V_2$ form the comparison system

$$\dot{\mathbf{z}}(t) = \mathbf{w}(\mathbf{z}(t)), \tag{37}$$

where $\mathbf{w}$ is defined by

$$\dot{z_1}(t) = -k_\alpha z_1(t)$$
$$\dot{z_2}(t) = L z_1(t) - k_\alpha z_2(t).$$

Lemma 10. Function $\mathbf{w}(\mathbf{x})$ of the comparison system satisfies the Kamke condition.

Proof. Write $\mathbf{w}(\mathbf{x})$ as $\mathbf{w}_1(x_1, x_2) = -k_1 x_1$ and $\mathbf{w}_2(x_1, x_2) = k_2 x_1 - k_3 x_2$, where k_1, k_2 and k_3 are positive constants. (i) Consider $\mathbf{w}_1$ and two arbitrary vectors $\mathbf{x}'$ and $\mathbf{x}''$ such that $x_1' = x_1''$ and $x_2' \leq x_2''$. $\mathbf{w}_1(\mathbf{x}') = -k_1 x_1'$ and $\mathbf{w}_1(\mathbf{x}'') = -k_1 x_1''$. Hence $\mathbf{w}_1(\mathbf{x}') = \mathbf{w}_1(\mathbf{x}'')$ and therefore $\mathbf{w}_1$ satisfies the Kamke condition. (ii) Consider $\mathbf{w}_2$ and two arbitrary vectors $\mathbf{x}'$ and $\mathbf{x}''$ such that $x_1' \leq x_1''$ and $x_2' = x_2''$. $\mathbf{w}_2(\mathbf{x}') = k_2 x_1' - k_3 x_2'$ and $\mathbf{w}_2(\mathbf{x}'') = k_2 x_1'' - k_3 x_2''$. Hence $\mathbf{w}_2(\mathbf{x}') - \mathbf{w}_2(\mathbf{x}'') = k_2(x_1' - x_1'') \leq 0$, therefore $w_2(\mathbf{x}') \leq w_2(\mathbf{x}'')$ and $\mathbf{w}_2$ satisfies the Kamke condition. Combining (i) and (ii), $\mathbf{w}(\mathbf{x})$ satisfies the Kamke condition. □

Lemma 11. The comparison system is globally exponentially asymptotically stable with equilibrium point $\mathbf{z} = \mathbf{0}$.

Proof. Comparison system (37) is a first-order linear differential equation with constant coefficients,

$$\dot{\mathbf{z}}(t) = \mathbf{A}\mathbf{z}^{\mathrm{T}},$$

where

$$\mathbf{A} = \begin{bmatrix} -k_\alpha & 0 \\ L & -k_\alpha \end{bmatrix}.$$

$\dot{z}_1(t) = 0$ implies $z_1 = 0$ and $\dot{z}_2(t) = 0$ implies $z_2 = 0$. According to the Routh-Hurwitz conditions (Murata (1977), p. 92) a 2×2 real matrix $\mathbf{A}$ is stable if and only if $\mathrm{tr}(\mathbf{A}) < 0$ and $\det(\mathbf{A}) > 0$. We have $\mathrm{tr}(\mathbf{A}) = -2k_\alpha < 0$ and $\det(\mathbf{A}) = k_\alpha^2 > 0$. $\mathbf{A}$ is therefore a stable matrix with negative eigenvalues. Stability properties are global for linear systems (Brauer and Nohel (1989), p. 151–152). The comparison system is therefore globally exponentially asymptotically stable. □

X.4. Stability of the simple production system

Theorem 12. The simple production system is asymptotically stable with equilibrium point $\mathbf{x} = [\mathbf{p}^*, \mathbf{q}^*]$.

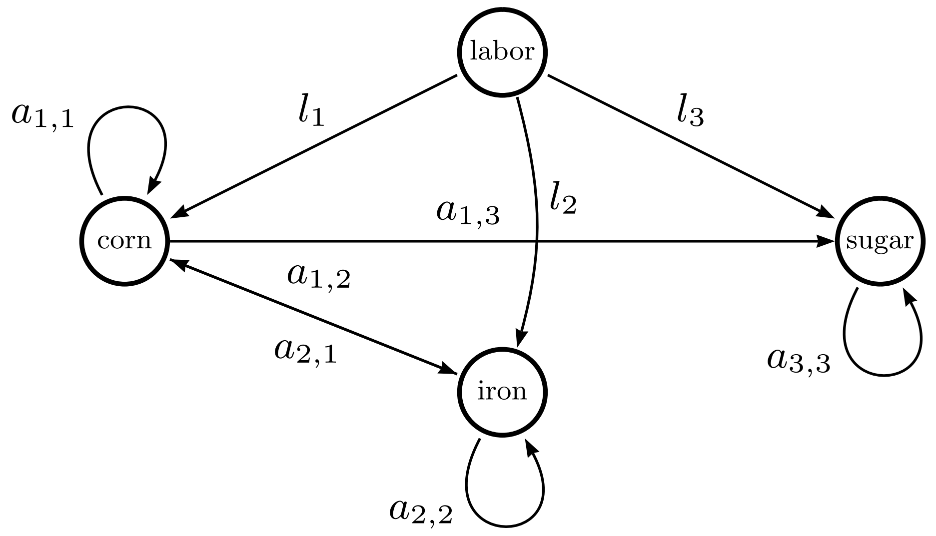
Proof. Candidate vector Lyapunov function (36) is zero at equilibrium, $\mathbf{V}([\mathbf{p}^*, \mathbf{q}^*]) = [0, 0]$, and the scalar function $v([\mathbf{p}, \mathbf{q}]) = \mathbf{u}\mathbf{V}([\mathbf{p}, \mathbf{q}]) > 0$, for $[\mathbf{p}, \mathbf{q}] \in \mathcal{D} - \{[\mathbf{p}^*, \mathbf{q}^*]\}$. From equations (36) and (36) we have

$$\nabla \mathbf{V}([\mathbf{p}, \mathbf{q}])\mathbf{f}([\mathbf{p}, \mathbf{q}]) \leq\leq [-k_\alpha V_1, L V_1 - k_\alpha V_2]$$
$$= \mathbf{w}(\mathbf{V}([\mathbf{p}, \mathbf{q}])),$$

where $\mathbf{f}$ is defined by (14), $\mathbf{w}$ satisfies the Kamke condition (Lemma 10) and $\mathbf{w}([0, 0]) = [0, 0]$. The zero solution $\mathbf{z}(t) = [0, 0]$ of the comparison system $\dot{\mathbf{z}}(t) = \mathbf{w}(\mathbf{z}(t))$ is asymptotically stable (Lemma 11). All the conditions of Theorem 7 are therefore satisfied. Hence $\mathbf{V}$ is a vector Lyapunov function and the simple production system is asymptotically stable. □

Local asymptotic stability implies that trajectories that start in the domain of attraction converge to the natural price equilibrium. But whether the simple production system is globally stable on $\mathcal{D}$ is an open question. Numerical solutions of the equations are inconclusive in this regard. A proof of local stability is sufficient, however, for the purpose of formally analyzing convergence to natural prices.

As an aside, the equilibrium price vector and quantity vectors are the left-hand and right-hand eigenvectors of the square matrix $\mathbf{A}$. The cross-dual dynamics of the simple production system (and nearby variants) is therefore a locally convergent iterative algorithm for finding the eigenvectors (see also Flaschel (2010, pg. 388)). Since each sector



$$\begin{bmatrix} \mathbf{A} \\ \mathbf{l} \end{bmatrix} = \begin{bmatrix} a_{1,1} & 0 & a_{1,3} \\ a_{2,1} & a_{2,2} & 0 \\ 0 & 0 & a_{3,3} \\ l_1 & l_2 & l_3 \end{bmatrix}.$$

FIG. 5. **Production graph for a 3-sector economy**. The graph describes the direct input requirements for the production of 1 unit of output. For example, 1 unit of corn directly requires $a_{1,1}$ units of corn, $a_{2,1}$ units of iron, and l_1 units of labor for its production.

adjusts to local information it should be possible to implement a fast, parallel implementation of the algorithm, which may have wider applicability than economic modeling.

XI. A NUMERICAL EXAMPLE

The equilibrium is stable but what does the process of gravitation toward equilibrium look like? Consider the following numerical example of a small, 3-sector economy that produces corn, sugar and iron, depicted in Figure 5. The parameters are:

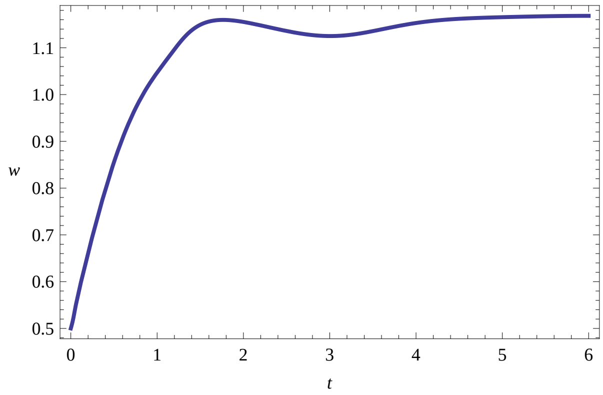
$$\mathbf{A} = \begin{bmatrix} 0.2 & 0 & 0.4 \\ 0.2 & 0.8 & 0.1 \\ 0 & 0 & 0.1 \end{bmatrix},$$

$\mathbf{l} = [0.7, 0.6, 0.3]$, $\mathbf{w} = [0.6, 0, 0.2]$ (i.e., workers consume corn and sugar but not iron), $\mathbf{p}(0) = [1.0, 0.8, 0.5]$, $\mathbf{q}(0) = [0.01, 0.1, 0.1]$ (i.e., the scale of corn production is initially relatively low), $\mathbf{s}(0) = [0.01, 0.1, 0.25]$ (i.e., the stock of corn inventory is initially relatively low), $w(0) = 0.5$, $m(0) = 1$ (i.e., the total money stock is 1), $\alpha = 0.8$ (i.e., workers tend to spend 4/5 of their savings), $L = 1$ (i.e., the total labor force is 1), $\lambda_1 = \lambda_2 = \lambda_3 = 1$, $\lambda_q = 0.25$ and $\lambda_w = 0.8$.

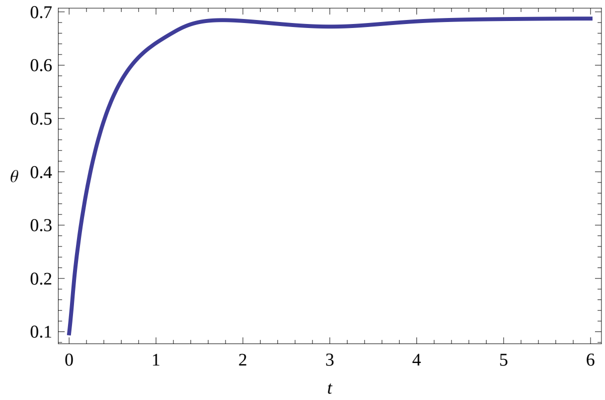
Figure 6 graphs numerical solutions of the trajectories of different variables of interest. Figure 6(a) graphs the wage rate and Figure 6(b) graphs the level of employment. As the level of employment rises, the labor market tightens, and the wage rate also rises, until the equilibrium level of employment is reached. Figures 6(c), and 6(e) graph prices and inventories respectively. The corn sector is of special interest. Initially, the scale of corn production and the stock of stored inventory is relatively low. The demand for corn (both directly from worker households and indirectly from the corn, iron and sugar sectors) exceeds the current supply. Soon after $t = 0$ the price of corn rises dramatically and the inventory plummets almost to zero. Figure 6(g) graphs each sector's profit (or loss) and shows that the increased price of corn initially increases the profits in the corn sector. Figure 6(d) graphs quantities. The initial high profits in the corn sector attract an inflow of capital and therefore an increase in the scale of production. At around $t = 1$ the supply of corn begins to meet demand and corn becomes progressively cheaper. In fact there is overshooting: corn production gets too high, supply exceeds demand and the corn sector begins to post a loss, causing a withdrawal of capital and a reduction in the scale of production. The iron and sugar sectors are simultaneously altering their prices and output quantities during this process.

The total stock of inventory (see Figure 6(e)) in general decreases over time. Figure 6(f) helps explain this trend. It graphs total employment subtracted by the labor-value of the real wage, which is the net 'inflow' of embodied labor to production from worker households. In this example the net 'inflow' is always negative. So the embodied labor withdrawn from production (in the form of the real wage consumed) in general exceeds the labor supplied to production (in the form of direct work performed). Recall that aggregate labor demand equation (12) equates the net inflow of labor to the change in the labor-value of inventories. So the stocks of inventories get depleted because worker households consume more than they produce, measured in terms of labor-time, during the process of gravitation.

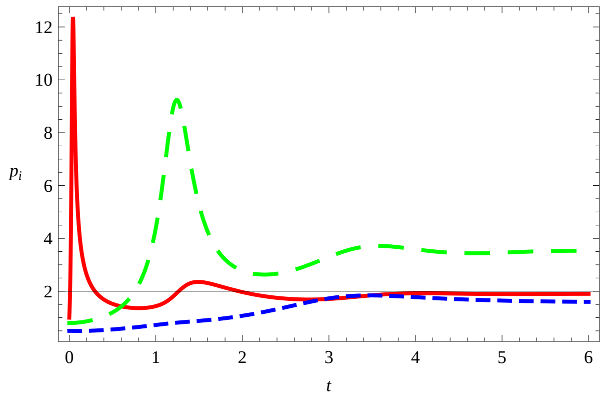
Overall, the demand for labor exceeds its supply, which causes an expansion of production and a rise in employ-



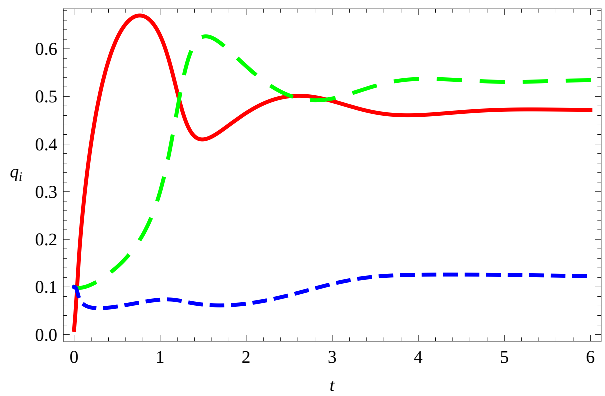
(a)Wage rate, w.



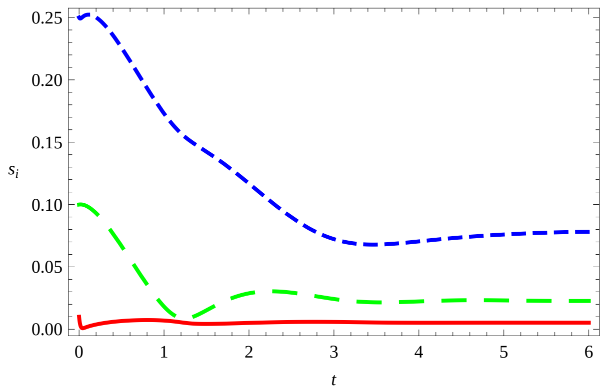
(b)Employment, $\theta = \mathbf{l}\mathbf{q}^{\mathrm{T}}$.



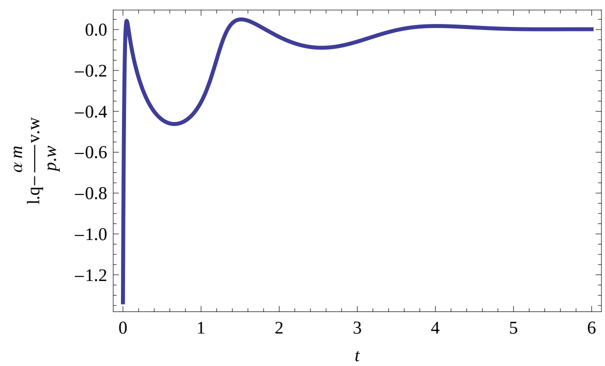
(c)Prices, $\mathbf{p}$.



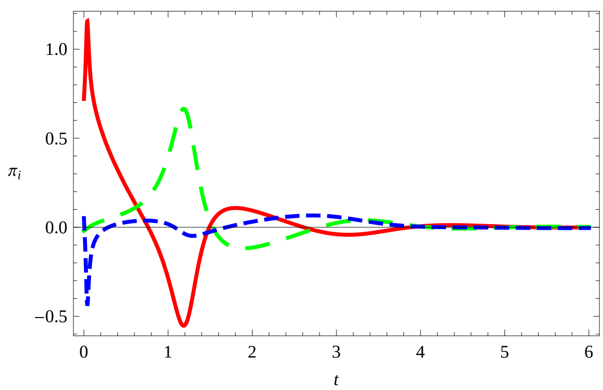
(d)Quantities, $\mathbf{q}$.



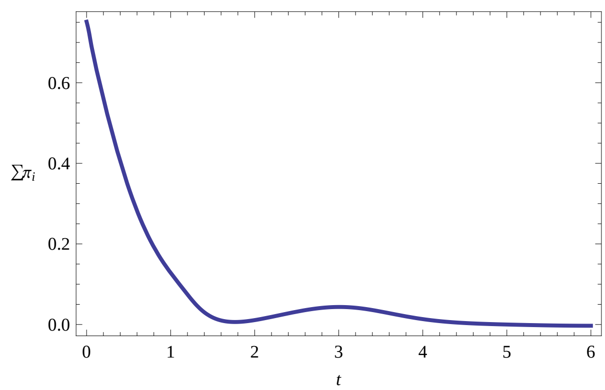
(e)Inventories, $\mathbf{s}$.



(f)Labor supply subtracted by the labor-value of the real wage, $\theta - \frac{\alpha m}{\mathbf{p}\mathbf{w}^{\mathrm{T}}}\mathbf{v}\mathbf{w}^{\mathrm{T}}$.



(g)Profit, π_i.



(h)Total profit, $\sum \pi_i$.

FIG. 6. Trajectories of an instance of a 3-commodity system. Corn is solid line, iron is dashed line, sugar is finely dashed line.

ment and wages. The physical net product initially does not match either the scale or composition of the real wage. So prices adjust to ration commodities in short supply, which causes profit rate differentials. Capital gets reallocated, which causes quantity adjustments. The process of cross-dual adjustment tends to eliminate supply and demand imbalances. The total profit decreases over time (see Figure 6(h)) until it reaches zero, at which point the aggregate demand equals the aggregate supply, all sectors are in balance, and prices are proportional to labor-values.

XII. SUBSTANCE AND FIELD

The classical proposition that 'long-period' equilibrium prices are proportional to labor-values in circumstances of simple production is not controversial; indeed, even critics of the labor theory of value accept this (e.g., Samuelson (1971); Steedman (1981); Roemer (1982)). But formal analysis of the labor theory are almost always formulated in terms of static equilibrium models where the question of convergence is either assumed or ignored. The model presented in this paper, together with an earlier stochastic agent-based model (Wright, 2008), clearly demonstrate that convergence is not an arbitrary assumption but a necessary consequence of market dynamics over reproducible commodities. The proposition that natural prices are attractors for market prices in conditions of simple production should therefore be considered relatively well-established. Of course, this does not imply that, in reality, actual market prices in fact realize their natural prices. Turbulent and ceaseless technical change continually moves the attractor before the economy has time to fully converge.

Most critics reject the labor theory based on the existence of a transformation problem between labor-values and prices of production, which arises in circumstances of simple reproduction where profit-income is distributed to a capitalist class. This paper has little to say about this controversy (but see Wright (2009) for a rebuttal). Mirowski (1989), however, offers a critique of Marx's value theory that applies in circumstances of simple production. We will examine his criticisms in order to demonstrate an application to value theory of the formal model developed in this paper.

XII.1. The 'swan song' of classical substance-based theories of value?

Mirowski's *More Heat Than Light* (1989) is one of the most stimulating and thought-provoking modern treatises on the history and theory of economic value. Mirowski draws out and critically examines the deep connections between modern economic theory and the physical sciences, especially with regard to conservation principles. Marx is accorded a special place in Mirowski's history.

Mirowski (1989, pgs. 174–185) claims that 'Marx simultaneously argued for two contradictory versions of the labor theory of value: the first of which we shall call the crystallized-labor or substance approach; the second is called the real-cost or virtual approach.' The 'substance approach' is the proposition that labor gets 'embodied' or 'crystallized' in its product, so every commodity is a carrier of an amount of an underlying labor substance. On the other hand, the 'real-cost approach' is the proposition that a commodity 'can only be said to possess a labor value in relation to the contemporary configuration of production. Although its physical complexion or its past history might persist unaltered, its real-cost labor value would be subjected to change by technological alterations anywhere in the economy' (Mirowski, 1989, pg.181). Mirowski claims the 'real-cost approach' is 'in direct contradiction to the crystallized approach', repeating an earlier argument by Cohen (1981) who also emphasized this dichotomy. It is worth quoting Mirowski at length in order to understand why he arrives at this conclusion.

> 'A clear example of the real-cost labor theory is provided by Marx's discussion of the effects of a harvest failure upon the existing stocks of cotton harvested in the previous year. In this passage he insists that a harvest failure would instantaneously revalue the embodied labor value of the cotton inventories in an upward direction, under the reasoning that the 'socially necessary' amount of labor-time to produce a bale had risen. This discussion stands in stark contrast to what would happen in a regime of crystallized values: There the cotton inventories would undergo no revaluation, even though the newly harvested cotton would.' (Mirowski, 1989, pg. 181).

Mirowski localizes the labor substance in the physical body of the commodity; for example, he writes that a 'chief characteristic' of a substance theory of value is 'the external residence of value in the commodity' (Mirowski, 1989, pg. 399). Clearly a change of labor productivity in cotton production cannot alter the amount of the labor substance already embodied in existing cotton inventories, unless we admit the existence of a mysterious kind of 'action at a distance'. Mirowski wonders why Marx would 'commit this blunder' since it means his theory suffers from a 'crippling problem'. Mirowski concludes, therefore, that Marx's work represents the terminus or 'swan song' of classical substance-based theories of value.

XII.2. Marx's 'social substance'

Is Mirowski correct in his assessment? The short answer is 'no' since Mirowski misreads Marx's concept of substance. Marx explicitly contrasts his concept of substance to a physical concept; he writes,

> 'the value of commodities is the very opposite of the coarse materiality of their substance, not an atom of matter enters into its composition. Turn and examine a single commodity, by itself, as we will, yet in so far as it remains an object of value, it seems impossible to grasp it. If, however, we bear in mind that the value of commodities *has a purely social reality*, and that

> they acquire this reality only in so far as they are *expressions or embodiments* of one identical *social substance*, viz., human labor, it follows as a matter of course, that value can only manifest itself in the *social relation* of commodity to commodity' (Marx, 1954) (emphasis added).

Marx's concept of substance is therefore essentially different from Mirowski's. The labor substance 'has a purely social reality' and therefore cannot *physically* reside in commodities (and how could a 'labor substance' be physically present in a commodity?) Arthur (2005) notes that all English translations of Marx's use of *Darstellung* in Capital 'are defective in offering "embodiment" as the translation'. He instead suggests that the phrase 'labor is "presented there" in the value of the product' better captures the intended meaning. Marx is therefore inviting us to consider that labor-value is an objective property of a commodity that emerges from a social practice, specifically a system of generalized commodity production. For example, in the appendix on the 'value form' in the first edition of Capital, Marx writes, 'The fact that products of labor – such useful things as coat, linen, wheat, iron, etc. – are *values, definite magnitudes of value* and in general *commodities*, are properties which naturally pertain to them only *in our practical interrelations* and not by nature like, for example, the property of being heavy or being warming or nourishing' (Marx, 1994). Let's try to unpack this collection of subtle but important ideas.

In Marx's theory the 'social substance' is abstract labor, which is the expenditure of the labor-power of workers considered as a homogeneous mass of productive capacity (the labor 'that forms the substance of value is homogeneous labor, expenditure of one uniform labor-power' (Marx, 1954)). The concept of a 'labor substance' that 'congeals' and gets 'embodied' in a commodity is *equivalent* to the concept that every commodity has an objective cost measured in terms of labor-time. For example, under the theoretical assumption of equal exchange, Marx (1954, pg. 59) writes, 'The equations, 20 yards of linen = 1 coat, or 20 yards of linen are worth one coat, implies that the same quantity of value-substance (congealed labor) is embodied in both; that the two commodities have each cost the same amount of labor of the same quantity of labor-time'. Marx then immediately adds, 'But the labor-time necessary for the production of 20 yards of linen or 1 coat varies with every change in the productiveness of weaving or tailoring'. In what sense then is labor-value 'crystallized', 'embodied' or 'expressed' in the body of a commodity if the amount of the value-substance is sensitive to a change in productivity?

XII.3. Labor-value is a field property

An object with mass has a weight in virtue of its casual relations to the gravity field in which it is embedded. Weight cannot be found 'in' an object, no matter how closely we examine it; nonetheless weight is a measurable property of an object necessary to explain its motion. Although weight is a property of an individual mass the property is derived from the context in which the mass is placed. Change the surrounding gravity field, for example by transporting the object to the moon, and the very same mass has a different weight. Let's call this kind of property a 'field property'.

Marx (1954, pg. 62–63) explicitly draws an analogy with weight to illustrate how labor-value manifests in the social practice of exchange. The analogy can also serve to illustrate the nature of 'embodiment' of the value-substance. A commodity with use-value acquires a labor-value in virtue of its causal relations to a system of generalized commodity production in which it is embedded. Labor-values cannot be found 'in' commodities; nonetheless labor-values are measurable properties of commodities necessary to explain their 'motion'. If the technical conditions of production should change, for example due to a change in labor productivity, then labor-values also change. Labor-value, therefore, is also a field property: it is a property of a commodity derived from the economic context in which it is placed.

The formal model developed in this paper can help illustrate these ideas. The technology represented by input-output matrix $\mathbf{A}$ and labor vector $\mathbf{l}$ is a discrete field that partly defines the economic context in which commodities are produced. The labor-values of commodities are determined by the field; the formula $\mathbf{v} = \mathbf{l}(\mathbf{I} - \mathbf{A})^{-1}$ makes this relationship precise and computable. If the 'field' should change, such as a change in the productivity of labor (from $\mathbf{l}$ to $\mathbf{l}'$) then labor-values change (from $\mathbf{v}$ to $\mathbf{v}'$). The labor-value of existing inventories, $\mathbf{s}$, is immediately 're-evaluated' since it now costs a different amount of total labor-time to produce that collection of commodities (i.e., $\mathbf{v}'\mathbf{s}^{\mathrm{T}}$ rather than $\mathbf{v}\mathbf{s}^{\mathrm{T}}$). The labor-value of a commodity is *defined* in terms of the 'field'. So a causal agent is not required to perform the consequent 're-evaluation' since the change in the labor-value of inventories is a conceptual, not a causal, necessity.

A change in the productivity of labor also modifies the attractor of the economy. An attractor predicts the motion of a system but is (normally) not explicitly represented within the system. So although the 're-evaluation' of labor-values is immediate it only empirically manifests over time in the 'motion' of commodities, such as the movement of relative prices, which, in consequence of Theorem 12, begin to converge to the new set of labor-values.

Does a mass 'have' a weight? We say it does, even though 'weight' is a field property and, on deeper reflection, is a *relation* between a mass, a gravity field and the laws of Newtonian mechanics. The same is true for labor-value: it is a relation between a use-value, the productivity of labor, and the dynamic laws of motion of commodity production, i.e. the 'law of value' (Marx, 1954). In this sense the value-substance is 'crystallized', 'embodied' or 'expressed' in the body of a commodity, and therefore we say that a commodity 'is' or 'has' a labor-value. And this is also why the notion of 'gravitation' toward natural prices is an essential theoretical element of the classical labor theory of value: the social practice of commodity production instantiates dynamic laws that cause labor-values to be causally efficacious and therefore explanatory; that is, labor-values only count because the economy counts them. Marx in addition chastised his classical forebears for not asking why labor expenditure takes the form of an exchange-value attached to things and developed an account, in the first few chapters of Capital, of why an unplanned social division of labor necessarily manifests a value-form (Engelskirchen, ming; Brown, 2008).

We can perhaps now understand better why Marx (1954) writes of the 'phantom-like objectivity' of the value-substance. Labor-value is a property of a material structure or activity that has physical extent and spatial location. Commodities, therefore, are 'mere congealations of human labor' (Marx, 1954). But labor-value is not a physical property and therefore does not respect Mirowski's commonsense notions of physical causality (for the same conclusion, from the perspective of Dialectical Materialism, see Brown (2008)). A change in the amount of value-substance embodied in an existing commodity due to a change in the conditions of production no more requires 'action at a distance' than does the change in status of a married person to a divorced person due to a legal act that happens to occur many hundreds of miles away. Labor-values are an emergent property of a social practice and therefore have a 'social' not a 'physical' reality. Mirowski simply gets Marx's concept of substance wrong.

XII.4. Integration over a field

Mirowski (1989, pg. 177) recognizes that Marx's theory has an explanatory structure analogous to field theories in the physical sciences. But his physical interpretation of the value-substance prevents him from understanding both Marx's theory and modern formalizations of it. For example, the Leontief inverse in the standard equation for labor-values, as we have seen, can be expanded as a infinite series; that is,

$$\begin{aligned}\mathbf{v} &= \mathbf{l}(\mathbf{I}-\mathbf{A})^{-1}\\ &= \mathbf{l}\sum_{n=0}^{\infty}\mathbf{A}^n\\ &= \mathbf{l}(\mathbf{I}+\mathbf{A}+\mathbf{A}^2+\mathbf{A}^3+\dots).\end{aligned} \tag{38}$$

In the Marxian literature it is common to interpret each term in the infinite series as representing production that occurred at a particular 'date'. The infinite series then represents a 'process' that occurs in time. The first term, $\mathbf{lI}$, represents the direct labor-cost of the final output of unit commodities at $t = 0$; the second term, $\mathbf{lA}$, represents the direct labor-cost of producing the heterogeneous inputs used-up by each sector at $t = -1$ in order to produce unit commodities as output at $t = 0$; and so forth, back in 'time'. So this 'dated' interpretation describes a process that extends into the past, until, in the limit, all commodities are ultimately reduced to labor alone. For example, Sraffa writes that the labor embodied in a commodity is 'the sum of a series of terms when we trace back the successive stages of the production of the commodity' (Sraffa (1960), p.89); and Samuelson writes that 'the accuracy of this result can be verified by going back in time to add up the dead labor needed at *all* the previous stages' (Samuelson, 1971).

Mirowski is correct to insist that the 'dated' expansion must be interpreted as a instantaneous property of a method of production; that is, labor-values are functions of the prevailing technology 'field'. Indeed, Marx consistently used current, not historical, labor costs in his theoretical work (see Moseley for a survey of the textual evidence). But Mirowski states that 'contrary to many modern Marxist writers, this is definitely not the crystallized-labor approach, except under the most counterfactual of circumstances that there has been no change in the entire history of capitalism with regards to the means of production' (Mirowski, 1989, pg. 182). This follows since, for Mirowski, labor is 'poured' (Mirowski, 1989, pg. 183) into commodities and is conserved in their bodies through time, much like a container stores an amount of liquid. Equation (38) is therefore 'simply false' under a substance interpretation because past labor costs must have differed to those prevailing today.

But, as we have established, in Marx's theory the value-substance is not a physical substance stored in the body of a commodity. The reduction to 'dated' labor is indeed a counterfactual interpretation of the calculation of current day real-costs. Again, a field analogy can help: the electrostatic potential energy of a charged particle in a field can be defined as the work that must be done to move it from an infinite distance away to its present location in the field. Physicists have used this definition to elucidate the meaning of potential energy. But the definition does not imply in any way that the particle was in fact moved through an infinite distance (and how could a particle be moved through an infinite distance?). Labor-values and potential energy are similar in this respect: both are instantaneous properties of 'objects' in a 'field' that have mathematical representations in terms of integrals or sums over fields. The reduction to 'dated' labor does not imply a real process that occurs in historical time. For example, Sraffa (1960), who perhaps first introduced the dated interpretation of the series reduction, is always careful to place 'date' in scare quotes.

Mirowski's argument bears a family resemblance to that of Bose (1980) who argues that abstract labor cannot be the substance of value since the reduction of commodities to labor-costs can never eliminate a commodity residue. No matter how far we go back 'in time' we always find labor combined with commodity inputs. As Keen (2001) remarks, if non-labor inputs were entirely eliminated then some commodities would be produced with zero commodity inputs, 'or in other words, by magic'. Keen considers Bose's logic to be 'impeccable' and therefore concludes, with Bose, that economic value cannot be reduced to labor-time. Bose certainly presents an impeccably literal interpretation of the series representation of labor-value accounting.

The infinite series expansion can also be interpreted as counting the amount of coexisting labor that reproduces a commodity (see section IX.5). And under this interpretation there is simply no concept of time or dates at all.

Rather than being 'simply false', as Mirowski suggests, equation (38) is a well-defined measure of the total amount of labor 'embodied' in a commodity. The measure has been operationalized in empirical studies, not only in the Marxian literature, but also in the form of employment multipliers in the Leontief-inspired input-ouput literature (e.g., see ten Raa (2005)).

XII.5. Mirowski's physicalism

Mirowski critique fails because he misunderstands Marx's concept of substance. He identifies Marx's conservation of value principles with the conservation of a physical value-substance that gets transported around the economy stored in the body of commodities (Mirowski, 1989, pg. 143). So the value-substance, once embodied, cannot subsequently change due to technical revolutions. Since Marx requires it can Mirowski only sees contradiction.

> In the crystallized-labor approach, the value substance is necessarily conserved in exchange, with Marx adding the further stricture that value is conserved in the transition between productive input and the output. The value accounts are clear and straightforward, not the least because they conform to the previous pattern of classical political economy. When it comes to the real-cost approach, all of the above principles are violated in one or another trans-temporal phenomenon; and Marx was not at all forthcoming about what he intended to put in their place. If we let the mathematical formalism dictate what is conserved, then [the reduction to dated labor expression (38)] dictates that it should be the technology that is conserved, for that plays the role of the field in the formalism; but as Marxian economics, this is nonsense. (Mirowski, 1989, pg. 183)

Mirowski's contradictions and anomalies disappear once we understand Marx's concept of substance and the difference between local conservation of labor-value and global field changes. For example, in the simple production system (14) labor-values are fixed and the value-substance is conserved in its 'journey' from its source in living labor, via multiple productive transformations, until destroyed in the sink of consumption (c.f. the aggregate labor demand equation (4)). If we introduce a technical change, that is modify either of the 'field' variables $\mathbf{A}$ or $\mathbf{l}$, then immediately we alter the total labor-value of inventories, the real wage, and the total value-substance flowing in the economy etc. The technical innovation has introduced a discontinuity, an interruption to local conservation, which alters the total labor-value embodied in the system. But in this new regime the value-substance is locally conserved just as before (e.g., equation (4) continues to apply).

Marx, like Smith and Ricardo, is well aware that technical change occurs all the time. But to understand the dynamics of a complex system we first need to abstract to keep some elements fixed in order to analyze dynamics that occur at different time scales (and see Foley (2008) for a discussion of the distinctive methodological approach of classical political economy). The local conservation of a value-substance should be understood in this context. We can push the field analogy further: the magnitude of a flux passing through a surface depends on the surrounding field. If the field changes then so does the flux. But physicists do not therefore reject local continuity equations. They instead develop more complex models, which retain a kernel of local continuity, within the context of time-varying fields. In some respects, Marx's approach in Volume 1 to conservation and non-conservation follows this pattern: he initially assumes the (local) conservation of value only to later introduce a special causal agent, human labor-power, which breaks conservation and produces relative surplus-value. In my view, one of the attractions of the class of model developed in this paper is that it may be possible, perhaps for the first time, to formally analyze Marx's theory of surplus-value in a properly dynamic setting.

Mirowski's thesis is that the value theories of Quesnay, Smith, Ricardo and Marx are all 'manifestations of a single class of value theory' (Mirowski, 1989, pg. 143). But the concept of a physical substance belongs to the Physiocrats, not Marx. Mirowski's thesis therefore represents an inadequate understanding of the content and intent of Marx's theory. A better analysis of the relationship between Marx and his precursors, including the Physiocrats, is given by Marx himself, in his posthumous *Theories of Surplus Value* (Marx, 2000).

Mirowski forces Marx to choose between a prosaic substance *or* 'nascent' field theory of value in order to avoid a contradiction. But Marx, if we are prepared to read his text carefully, presents a remarkably sophisticated and consistent substance *and* field theory of value, aspects of which can be precisely formulated in the language of dynamic systems theory.

XIII. CONCLUSION

Marx, like Smith and Ricardo, shared a vision of a market economy as a dynamic system that tended to converge toward or gravitate around a natural price equilibrium. Marx called the tendency for market prices to gravitate toward natural prices the 'law of value'. This paper is a formal analysis of the law of value operating in the specific circumstances of simple production. In essence, any mismatches between supply and demand create arbitrage opportunities that attract capital, which functions to allocate the available social labor between sectors of production, such that the economy 'gropes' toward a scale and composition of output that equals demand, at which point prices are proportional to labor values.

The simple production system is just one variant of a whole family of cross-dual adjustment models. Many of the assumptions need to be relaxed in order to make truly general statements. Nonetheless the model developed in this paper is a powerful example that the classical theory of competition is a successful explanation of the homeostatic kernel of generalized commodity production.

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